Effective Pairings

in Isogeny-based Cryptography

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Pairings map
elliptic curve problems
to finite field problems

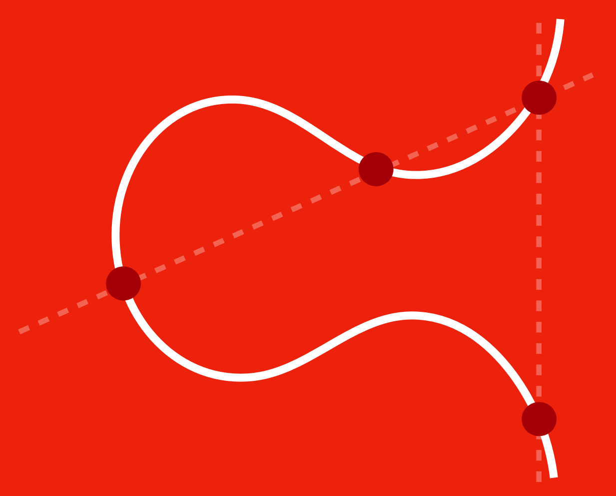
Elliptic curve arithmetic
is slow

Finite field arithmetic
is (very) fast

Hence, fast pairings
means fast solutions

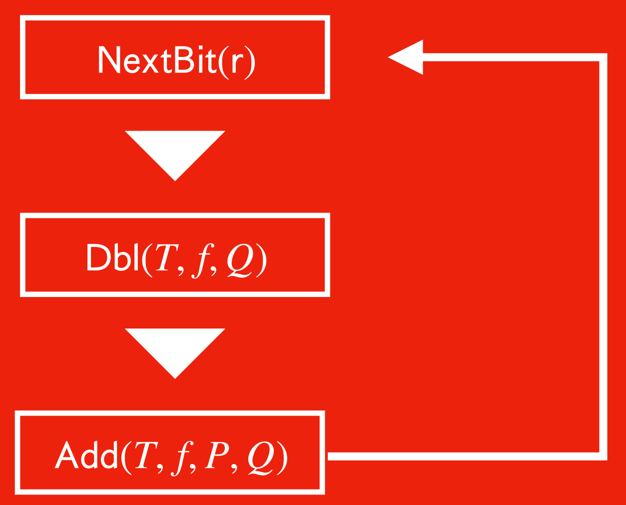
Effective Pairings in Isogeny-based Cryptography

1



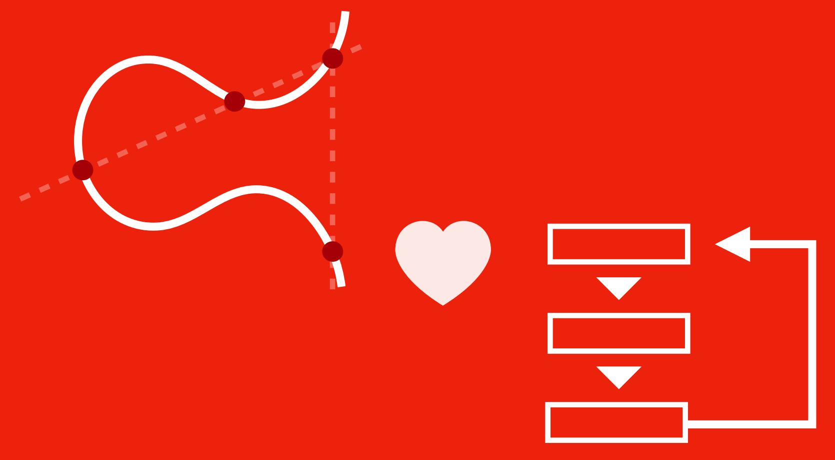
Isogenies & Pairings

2



Speeding-up general pairings

3



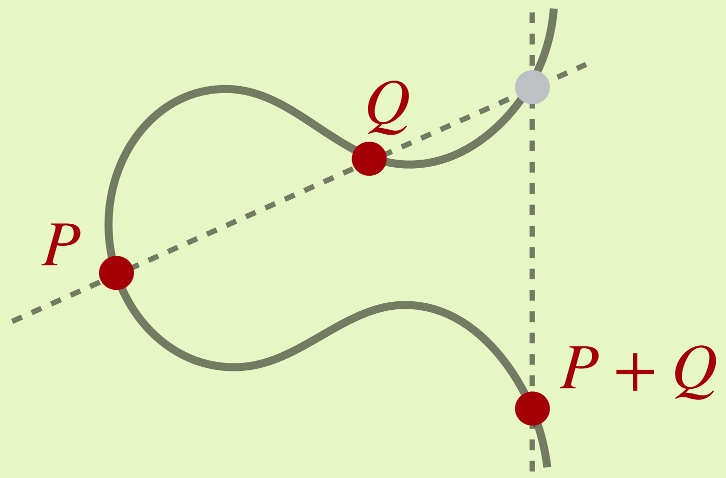
Applying pairings in isogeny crypto

What are pairings and what are isogenies?

elliptic curves in CSIDH

supersingular elliptic curve

- has $p + 1$ points in $E(\mathbb{F}_p)$
- choose p so that $p + 1 = 4 \cdot \ell_1 \cdot \ell_2 \cdot \ldots \cdot \ell_n$
- this implies the rational points on E have orders that divide $p + 1$



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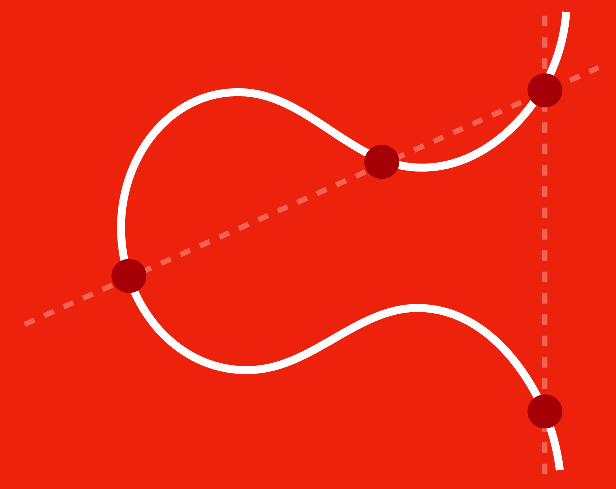

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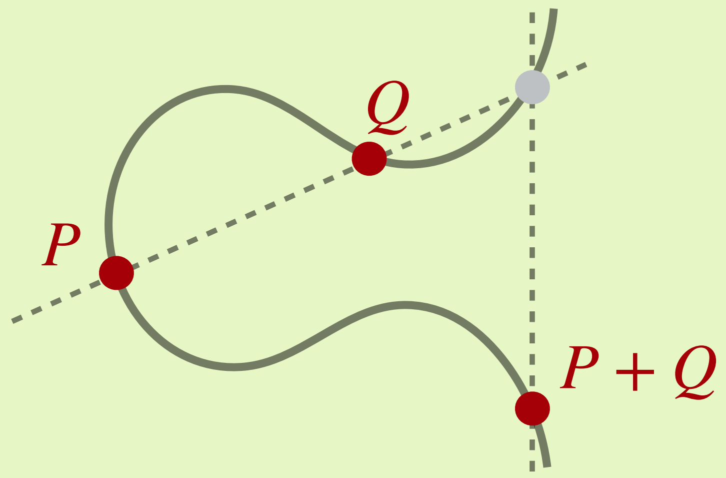


$$E : y^2 = x^3 + Ax^2 + x, \quad A \in \mathbb{F}_p$$

points on such curves

We have that

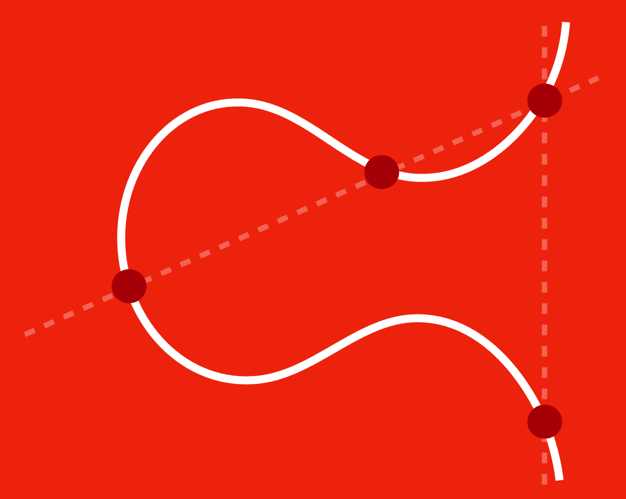
$$E(\mathbb{F}_p) \cong \mathbb{Z}_4 \times \mathbb{Z}_{\ell_1} \times \mathbb{Z}_{\ell_2} \times \ldots \times \mathbb{Z}_{\ell_n},$$

So think of a point $P \in E(\mathbb{F}_p)$ as a sum of points P_i of order ℓ_i

$$P = P_0 + P_1 + P_2 + \ldots + P_n$$

which shows how scalars $[\lambda]$ with $\lambda \in \mathbb{N}$ affect the torsion

$$\begin{aligned} [\ell_2]P &= [\ell_2]P_0 + [\ell_2]P_1 + [\ell_2]P_2 + \ldots + [\ell_2]P_n \\ &= [\ell_2]P_0 + [\ell_2]P_1 + \quad \mathcal{O} \quad + \ldots + [\ell_2]P_n \end{aligned}$$



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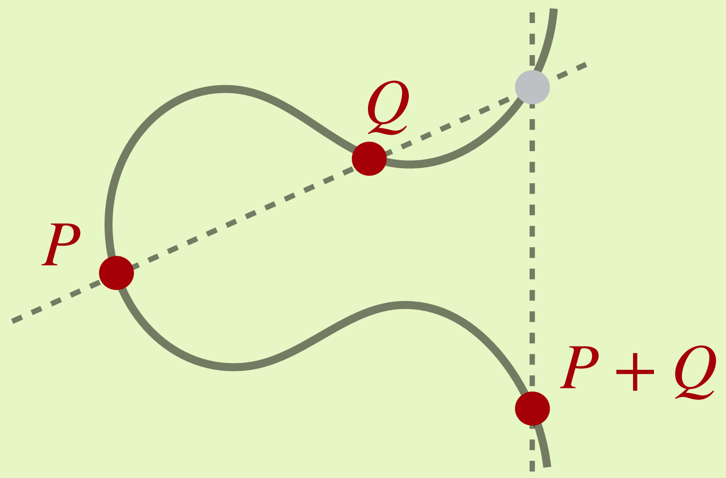

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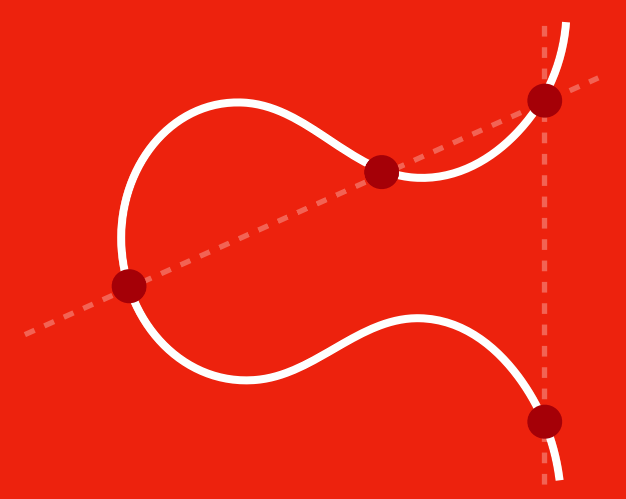
the order of P is readable from the non-zero P_i's

the torsion that P is *missing* are precisely the zero P_i's

full-torsion points

we call a point $P \in E(\mathbb{F}_p)$ a **full-torsion point** if the order is $p+1$, equivalently, all P_i are non-zero



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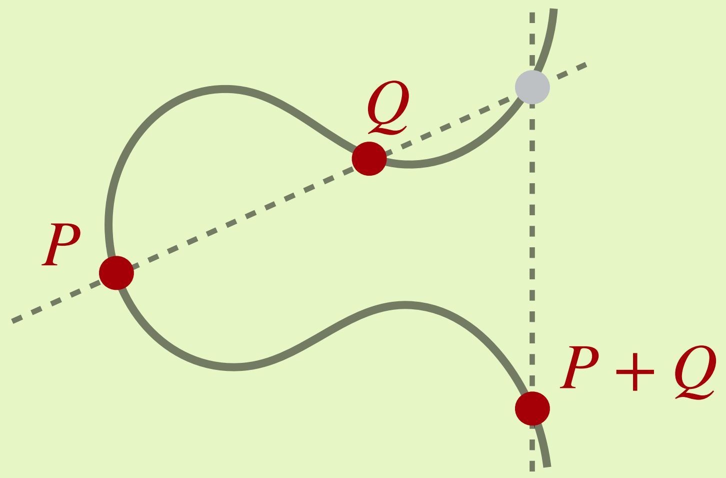

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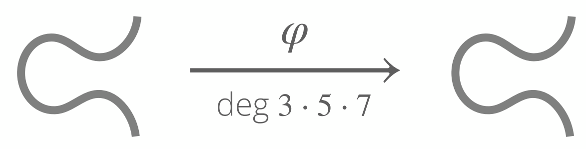
torsion points and isogenies

1. Any* isogeny φ of degree N
   - given by kernel of size N
   - generated by point P of order N



2. Any* isogeny φ of degree $N = \prod \ell_i$
   - splits into sub-isogenies of degree ℓ_i
   - each generated by point P of order ℓ_i

3. Any* isogeny φ of degree $N = \prod \ell_i$
   - computed using one **full-torsion** P
   - per ℓ_i, compute $[\frac{p+1}{\ell_i}]P$ to get $\ker(\varphi_i)$

$P = P_3 + P_5 + P_7 \in E(\mathbb{F}_p)$

$[5 \cdot 7]P = P_3' + \mathcal{O} + \mathcal{O} \in E(\mathbb{F}_p)$

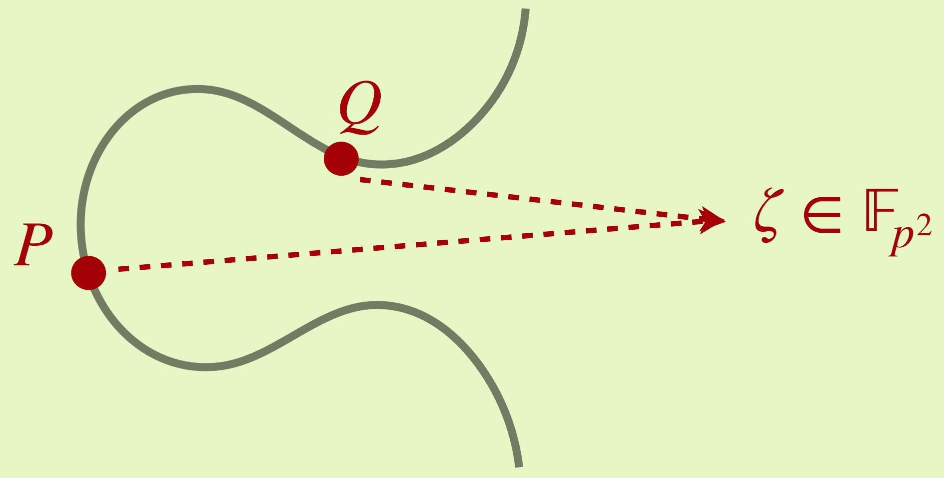
$\varphi_1(P) = \mathcal{O} + P_5' + P_7' \in E'(\mathbb{F}_p)$

*cyclic, separable

the Tate pairing*

bilinear pairing from torsion groups to fields

- choose a degree r
- take point P of order r on E, that is $P \in E(\mathbb{F}_{p^2})[r]$
- take point Q on E such that $Q \in E(\mathbb{F}_{p^2})/rE(\mathbb{F}_{p^2})$
- then $e_r(P, Q) = \zeta \in \mu_r$



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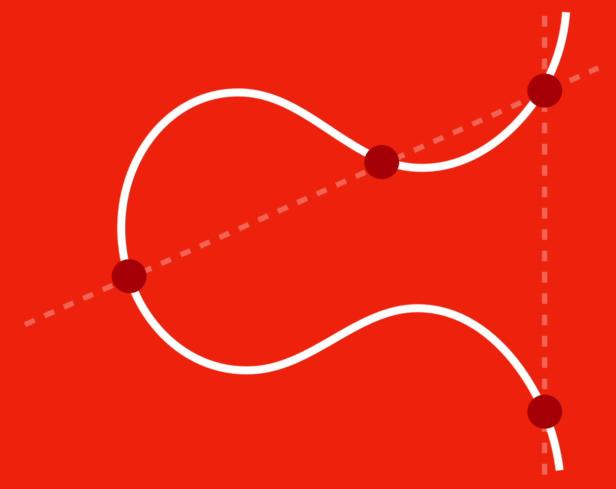

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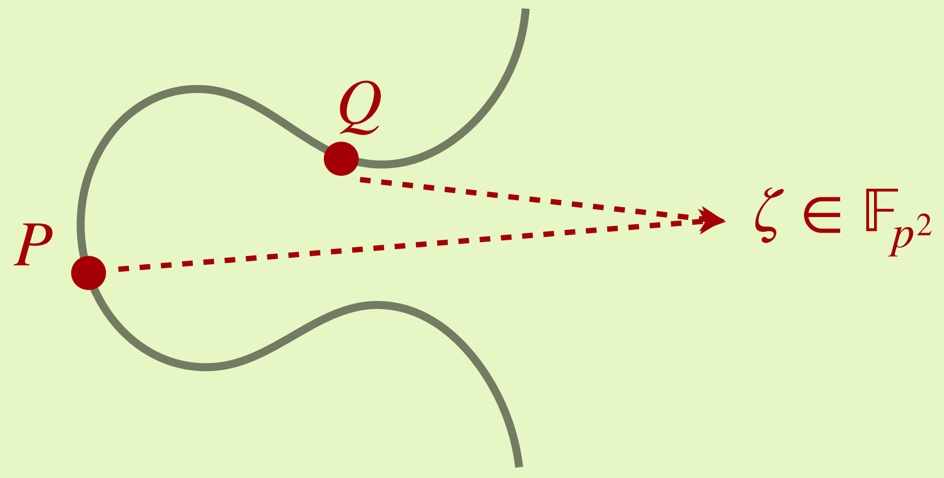

in our specific case

Formally, this pairing is abstract. Specifically in our case, $p+1 = 4 \cdot \ell_1 \cdot \ell_2 \cdot \ldots \cdot \ell_n$ there is a nice interpretation of this pairing.

Choose r dividing $p+1$, say $r = \prod \ell_i = \frac{p+1}{4}$ then for $P \in E(\mathbb{F}_p)$ we get

$$P = \mathcal{O} + P_1 + P_2 + \ldots + P_n .$$

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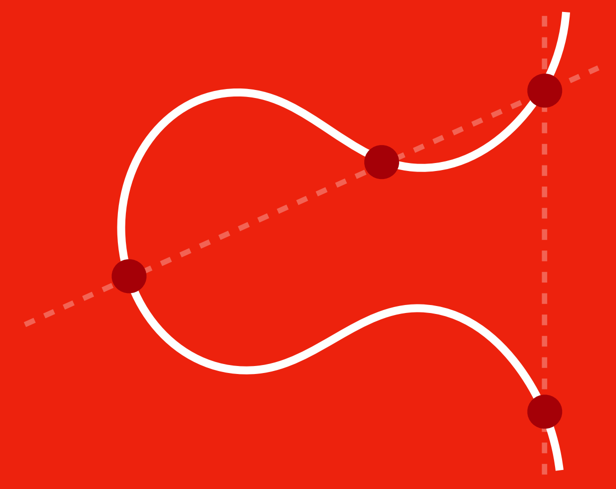

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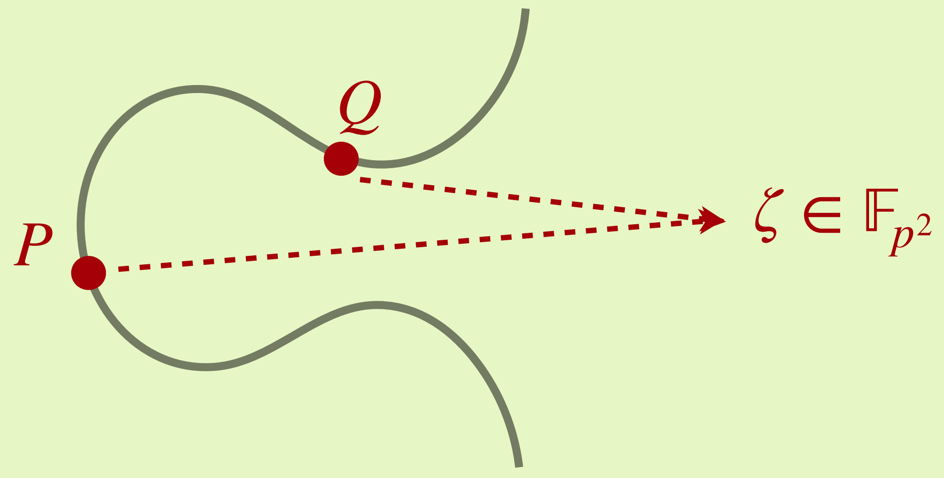

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For $Q \in E(\mathbb{F}_p)$, we have equivalence by elements R in $rE(\mathbb{F}_{p^2})$. In this scenario, we can think of such elements R as $R_0 + \mathcal{O} + \ldots + \mathcal{O}$, which implies $Q \sim Q'$ whenever

$$Q = Q_0 + Q_1 + Q_2 + \ldots + Q_n \quad \sim \quad Q' = Q_0' + Q_1 + Q_2 + \ldots + Q_n$$

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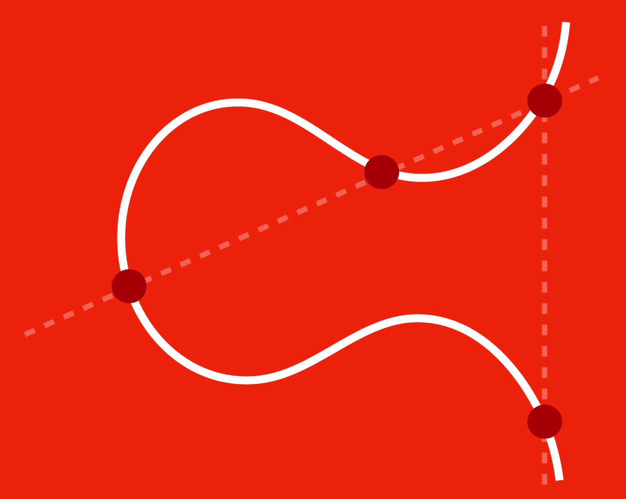

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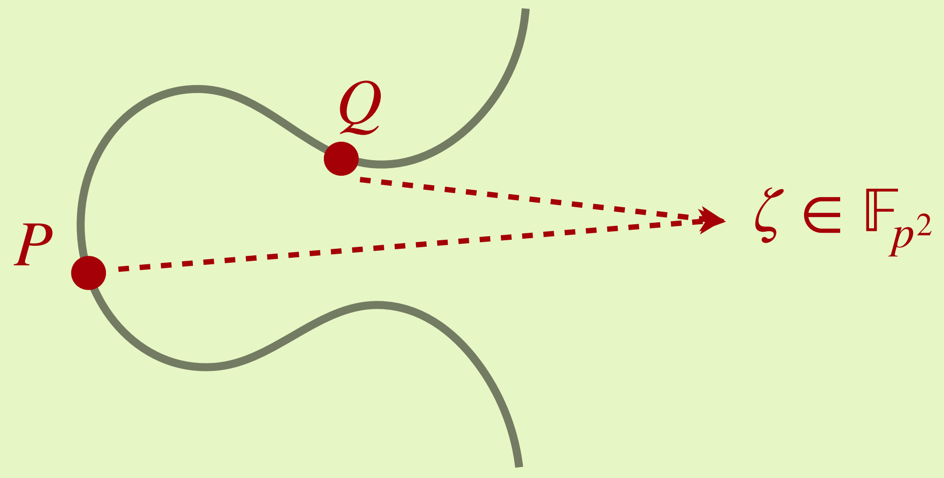

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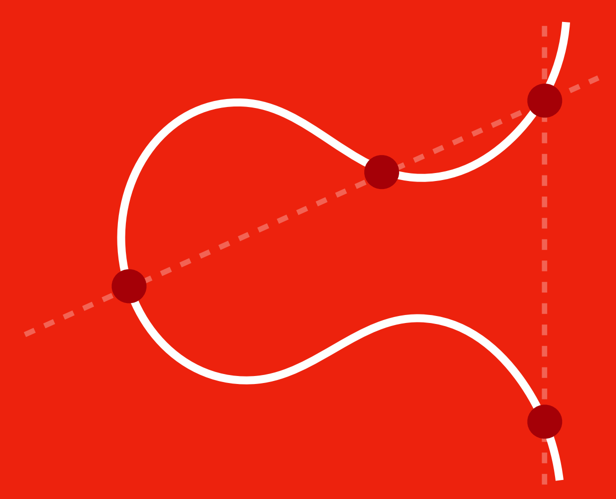
problem

If we pick $P, Q \in E(\mathbb{F}_p)$, then Q is a multiple of P. Then $e_r(P, Q) = 1$

*reduced



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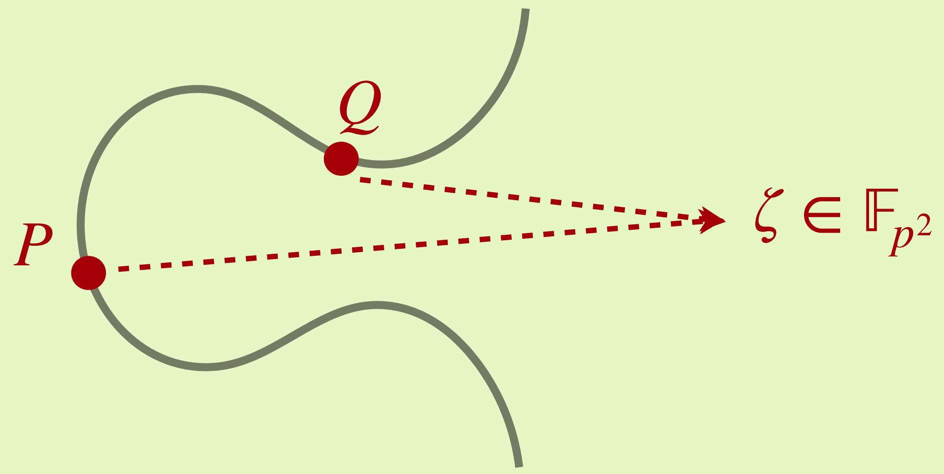

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If we pick $P, Q \in E(\mathbb{F}_p)$, then Q is a multiple of P. Then $e_r(P,Q) = 1$

✓ **solution**

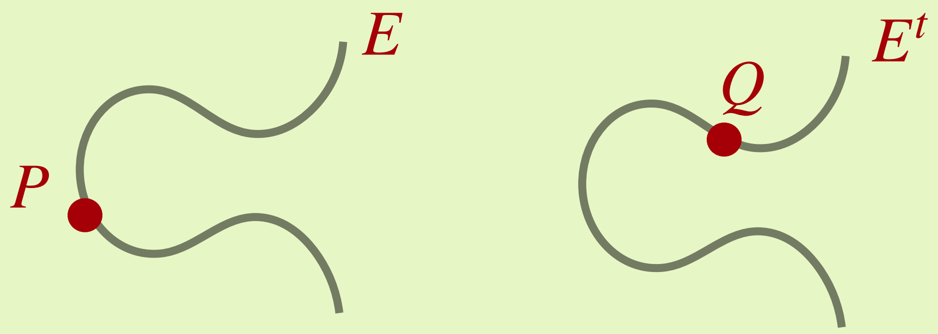
Work over $E[r] \subseteq E(\mathbb{F}_{p^2})$. In our specific case, just use Q on the *twist*

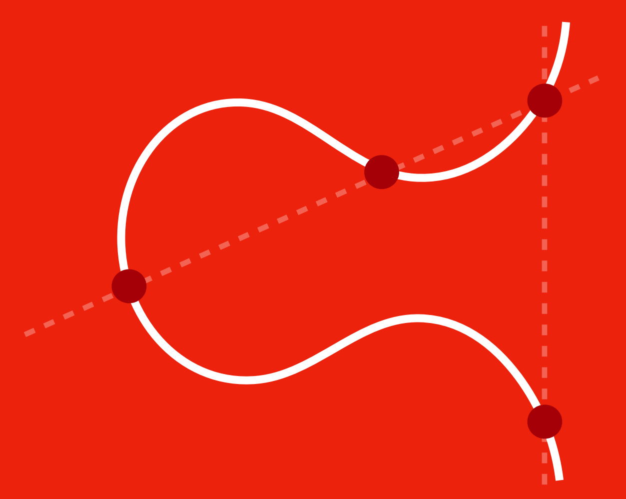
*reduced

the twist of E

Twist over $\mathbb{F}_p$ of supersingular curve E

- a curve E^t with $p + 1$ points over $\mathbb{F}_p$
- isomorphic to a specific subset of $E(\mathbb{F}_{p^2})$
- used in CSIDH to "move backwards" in graph
- want $P \in E(\mathbb{F}_p)$ and $Q \in E^t(\mathbb{F}_p)$, both full order





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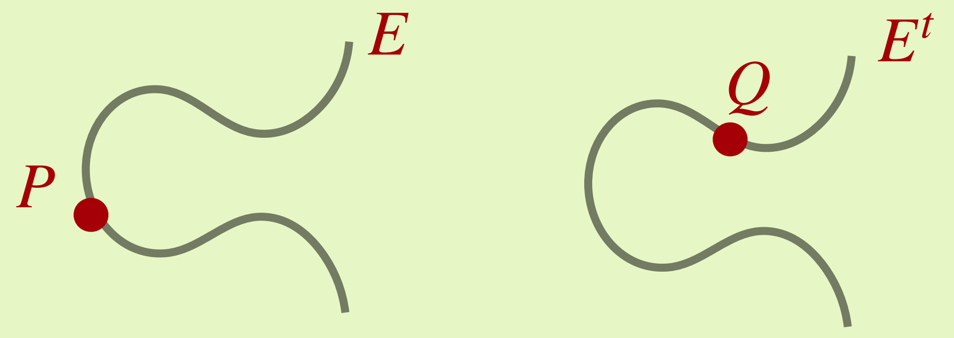

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consider P and Q as

$$P = P_0 + P_1 + \ldots + P_n$$
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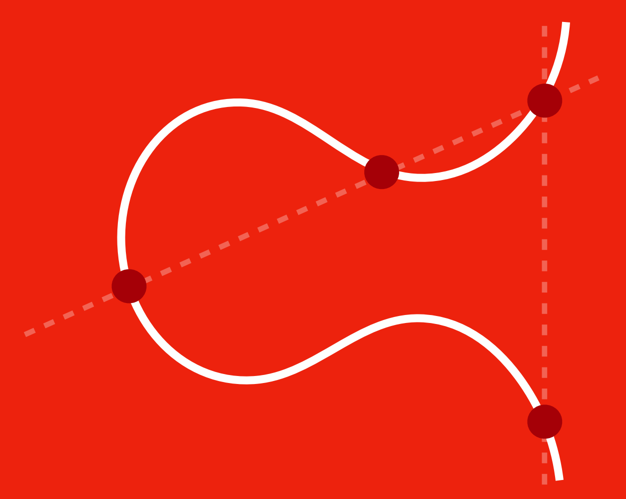
let $r = p + 1$

Tate pairing $e_r(P, Q)$ captures where **both** $P_i, Q_i \neq \mathcal{O}$



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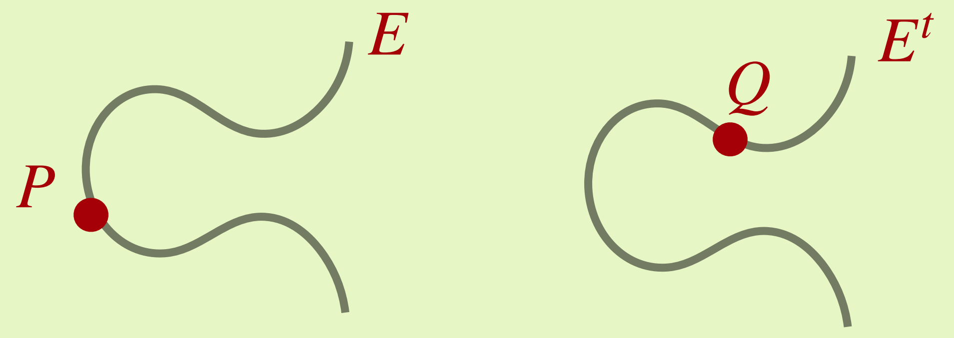

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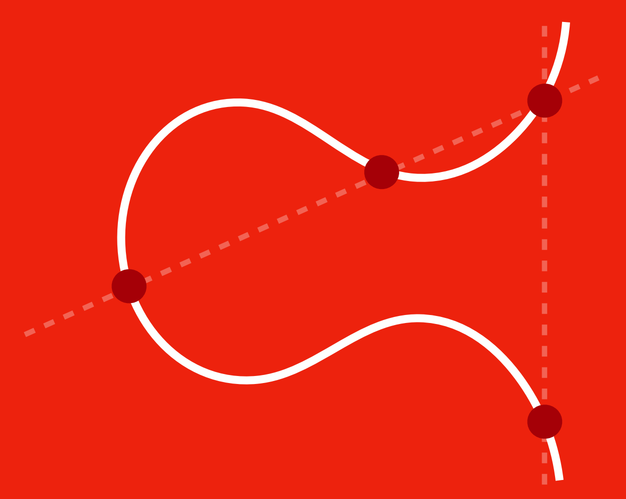
crucial lemma

Let $P \in E(\mathbb{F}_p)$, $Q \in E^t(\mathbb{F}_p)$, and $r = p+1$. Let $\zeta = e_r(P, Q) \in \mathbb{F}_{p^2}$.

Then ζ is an r-th root of unity, whose order is precisely $\gcd$ of order of P, order of Q



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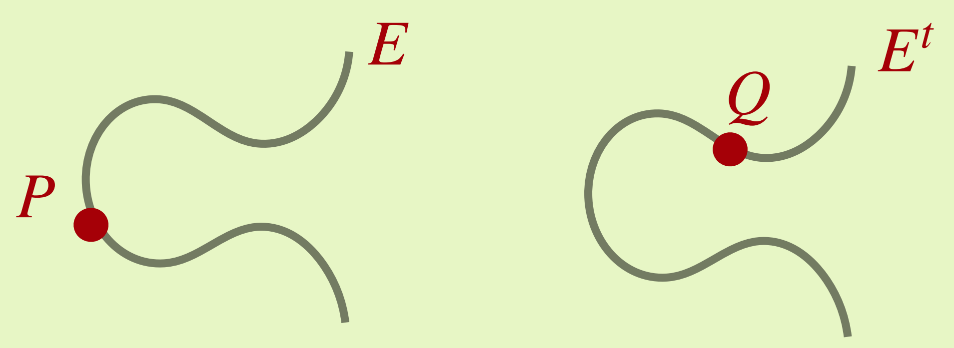

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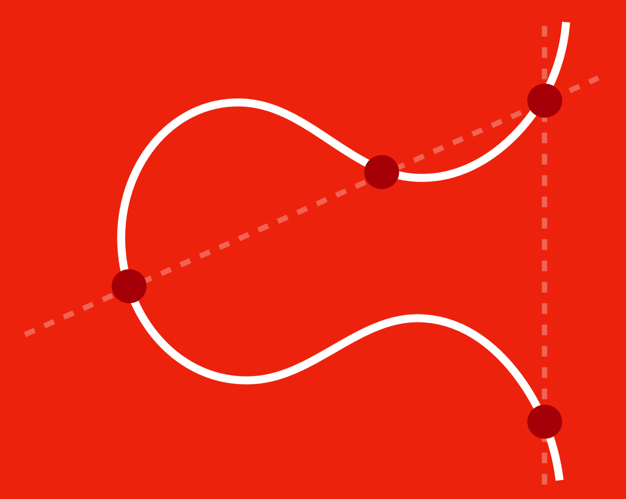
example

If P and Q both full torsion, then ζ has order $r = p+1$

example

If P has order 5, and Q has order 15, then ζ has order 5



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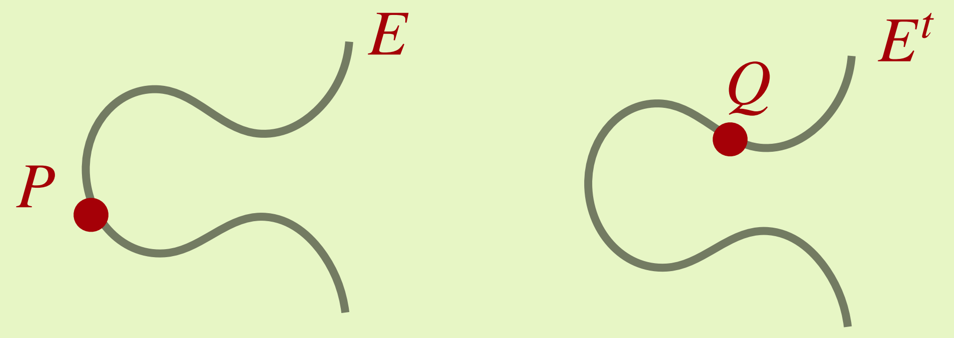

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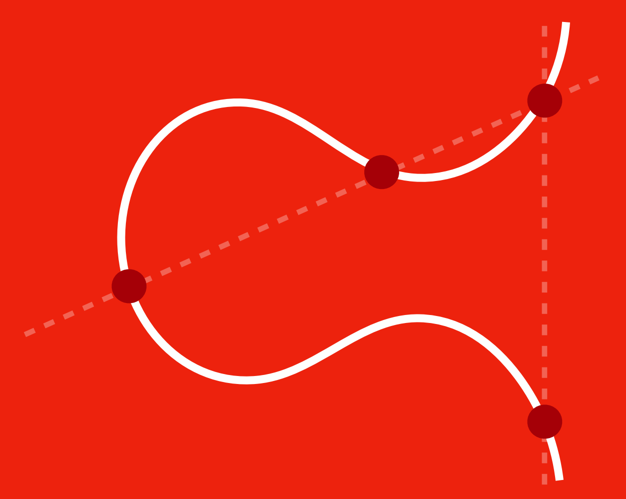
notice

Curve arithmetic is slow!
Field arithmetic is fast!!
(more than factor 6)



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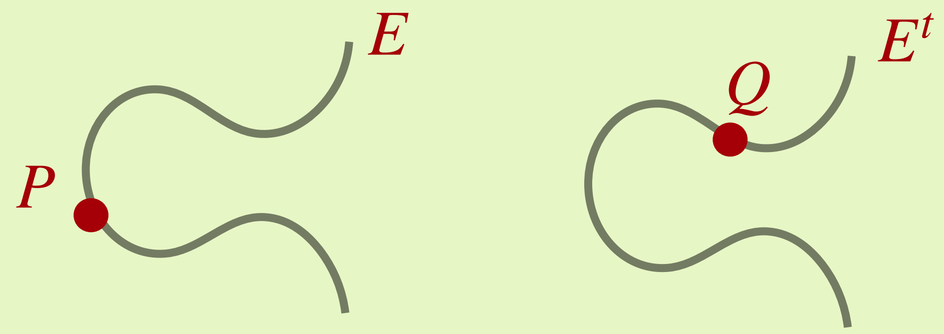

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let $r = p + 1$

Tate pairing $e_r(P, Q)$ captures where **both** $P_i, Q_i \neq \mathcal{O}$

crucial lemma

Let $P \in E(\mathbb{F}_p)$, $Q \in E^t(\mathbb{F}_p)$, and $r = p + 1$. Let $\zeta = e_r(P, Q) \in \mathbb{F}_{p^2}$.

Then ζ is an r-th root of unity, whose order is precisely $\gcd$ of order of P, order of Q

example

If P and Q both full torsion, then ζ has order $r = p + 1$

example

If P has order 5, and Q has order 15, then ζ has order 5

!

notice

Curve arithmetic is slow!
Field arithmetic is fast!!
(more than factor 6)

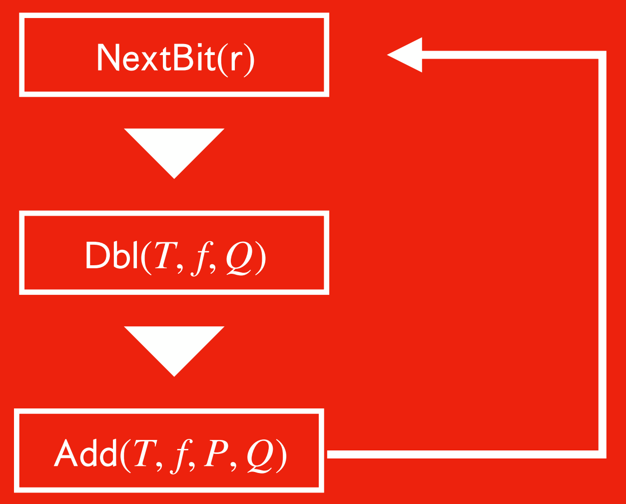
✓

core idea

Pick random $P \in E(\mathbb{F}_p)$ and $Q \in E^t(\mathbb{F}_p)$
Instead of using curve arithmetic to compute their orders, use ζ to compute the overlap in orders!

Pairings are quite slow

2



Speeding-up general pairings

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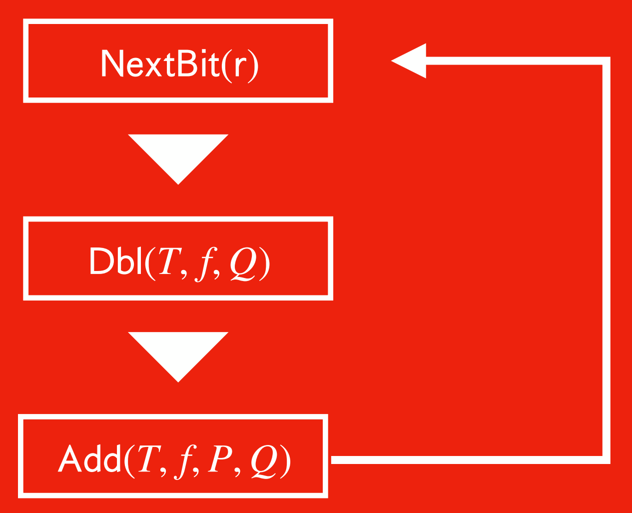

Speeding-up general pairings

pairing crypto

Choose a “nice” curve E,
Choose a “nice” prime p,
to do **pairings** with

Computing $e(P, Q)$
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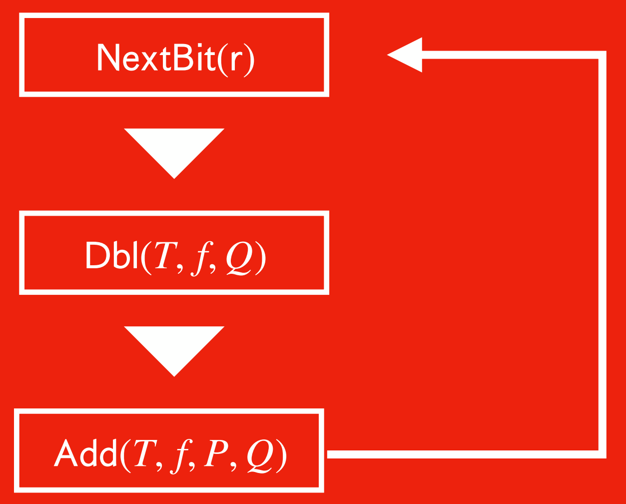

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These are mediocre curves,
and definitely bad primes,
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Computing $e(P, Q)$
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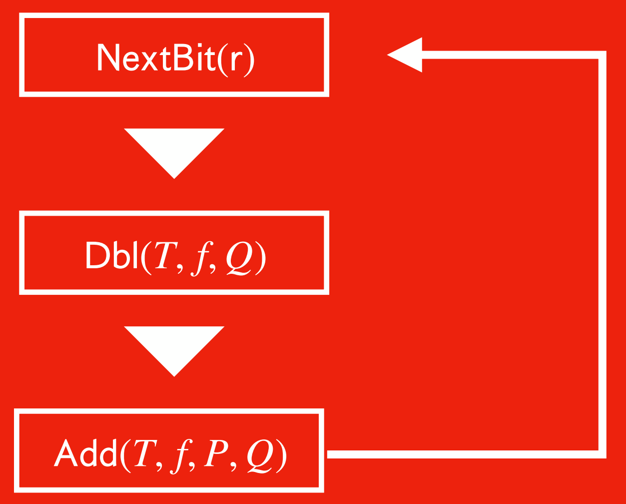

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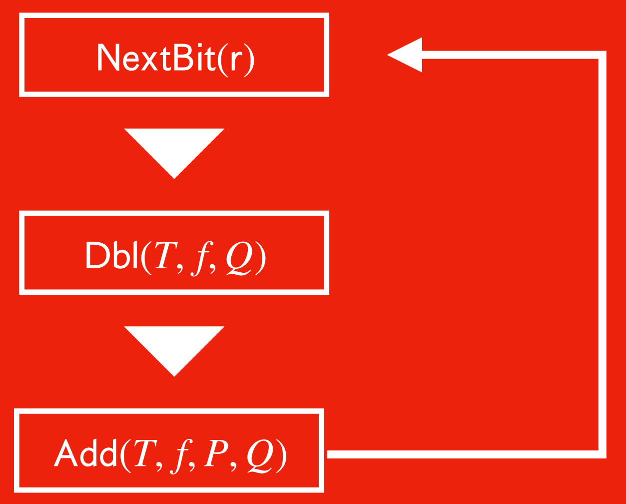

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MAIN RESULTS

1 make pairings great again

2 apply **core idea**

3 faster isogeny algorithms!

2

NextBit(r)

Dbl(T, f, Q)

Add(T, f, P, Q)

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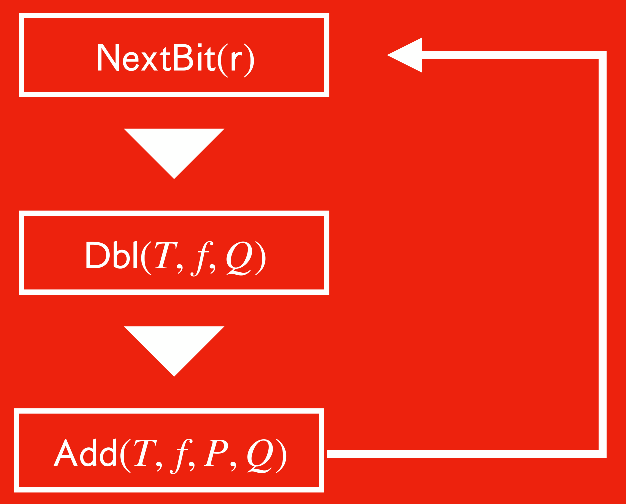

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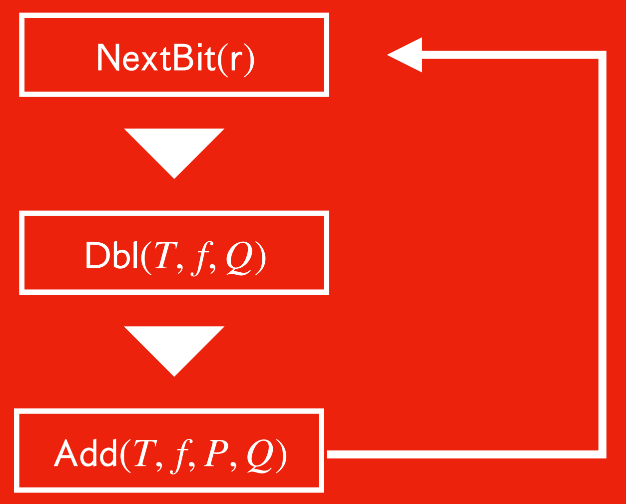

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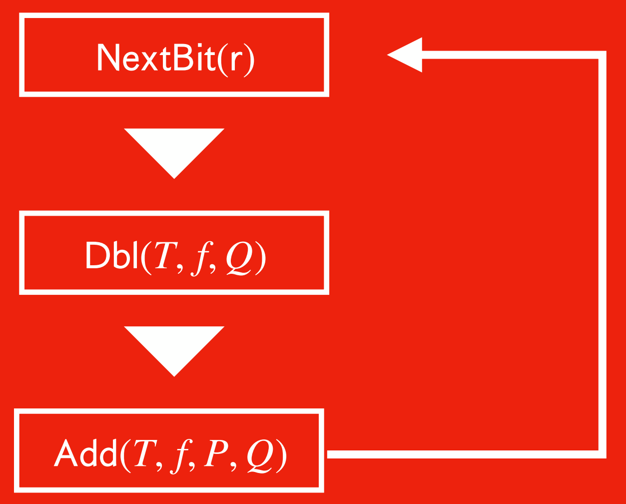

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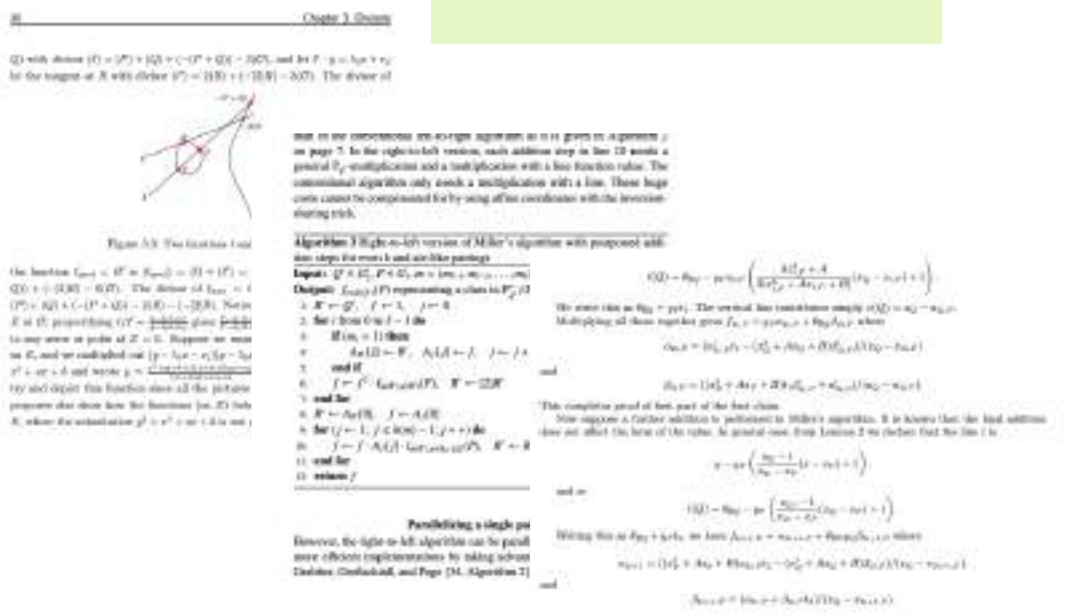

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implement all tricks that apply

2

benchmark speed and finetune

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fast pairings

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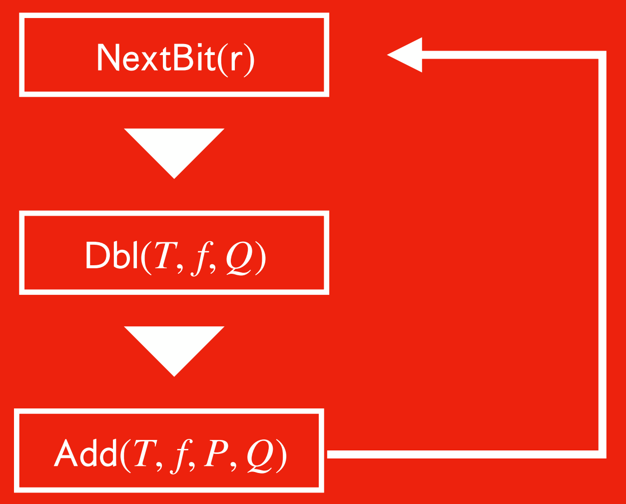

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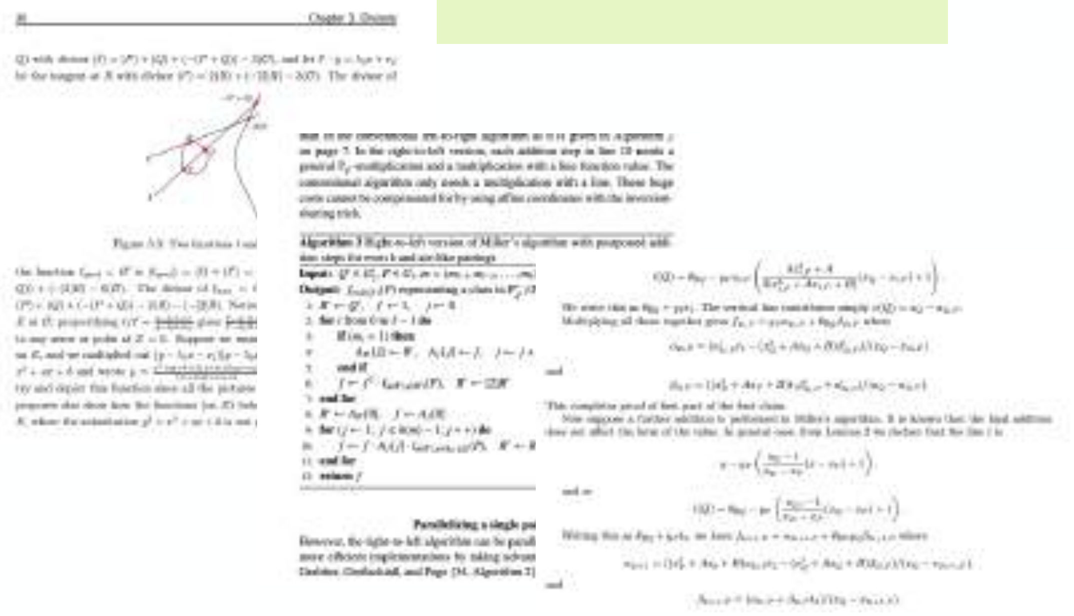

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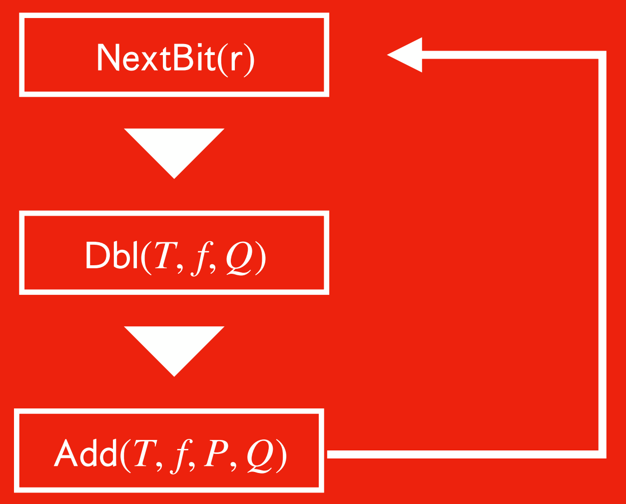

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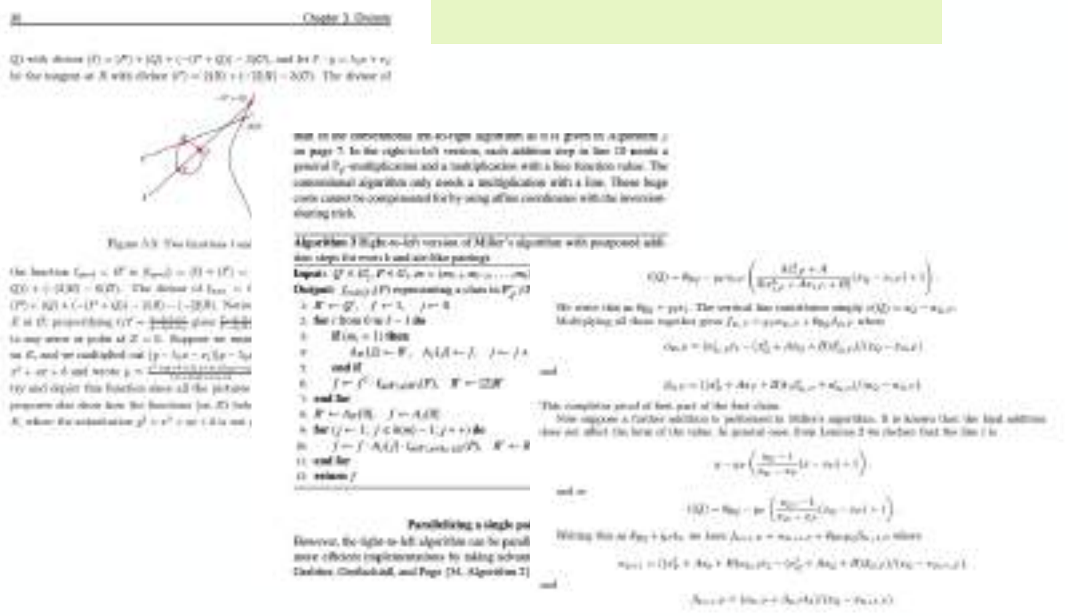

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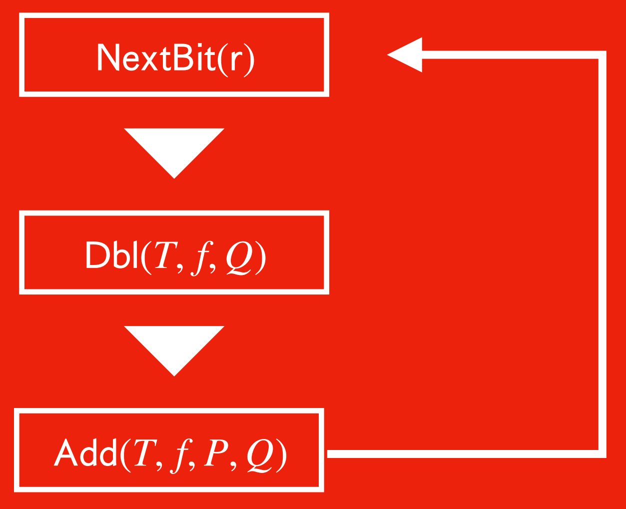

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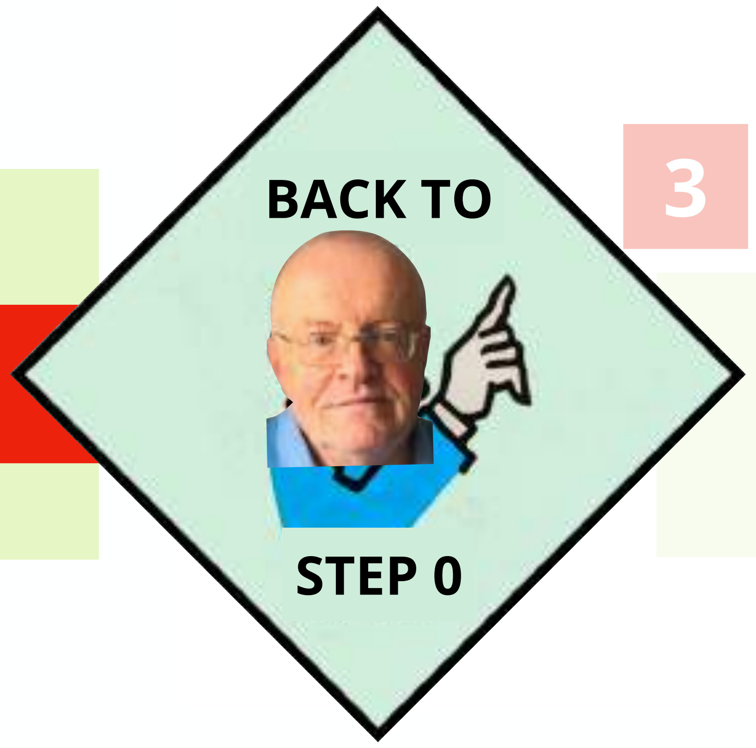

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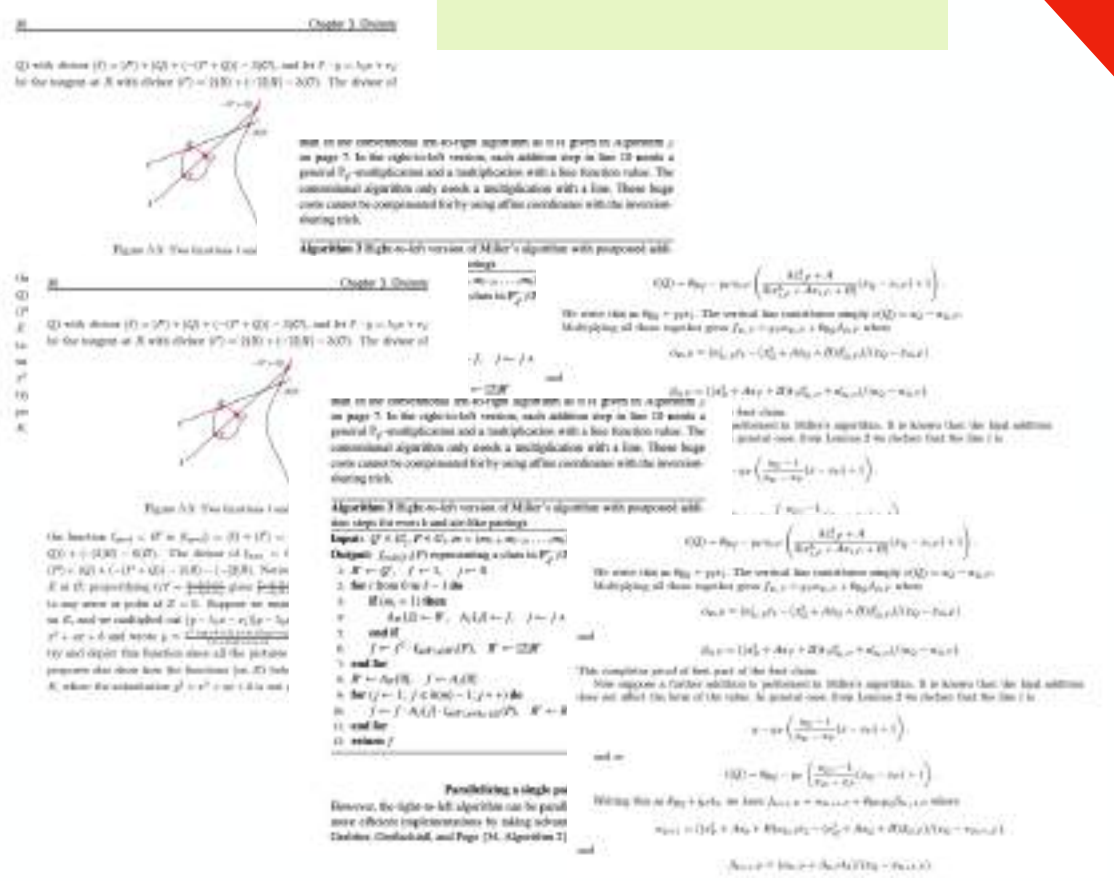

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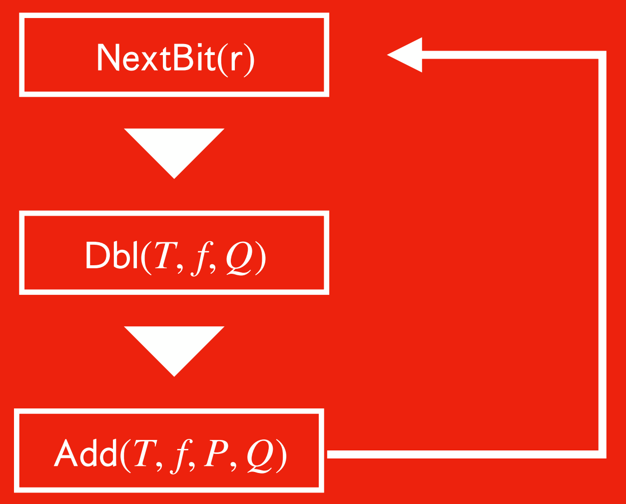

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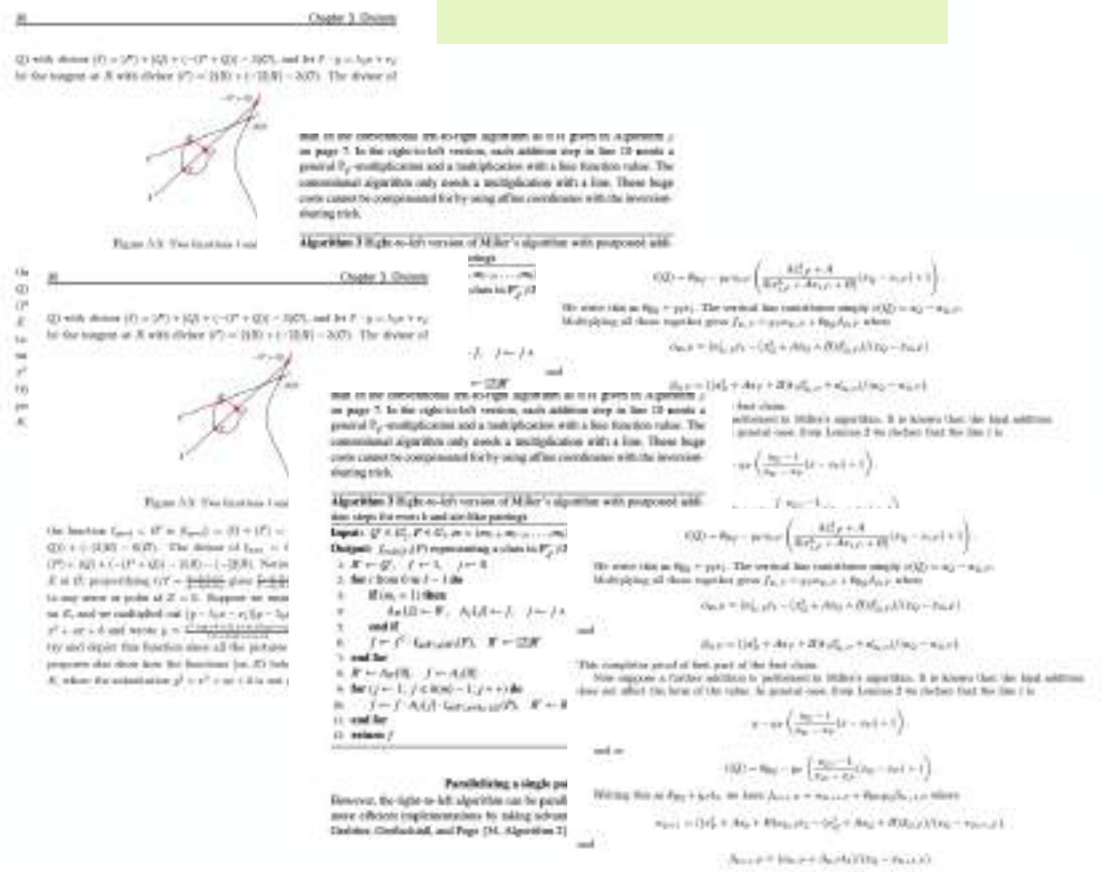

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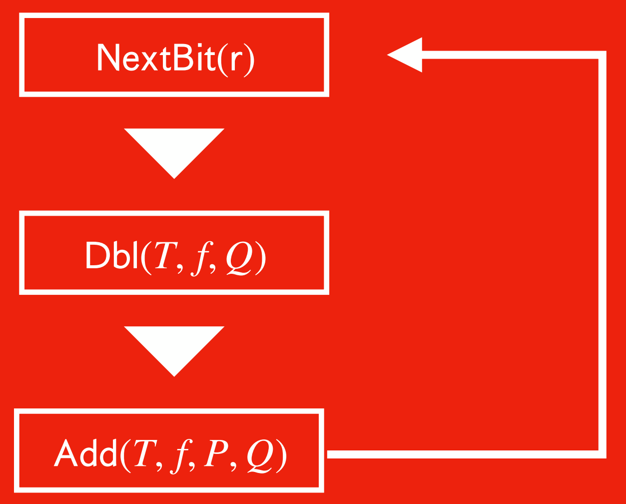

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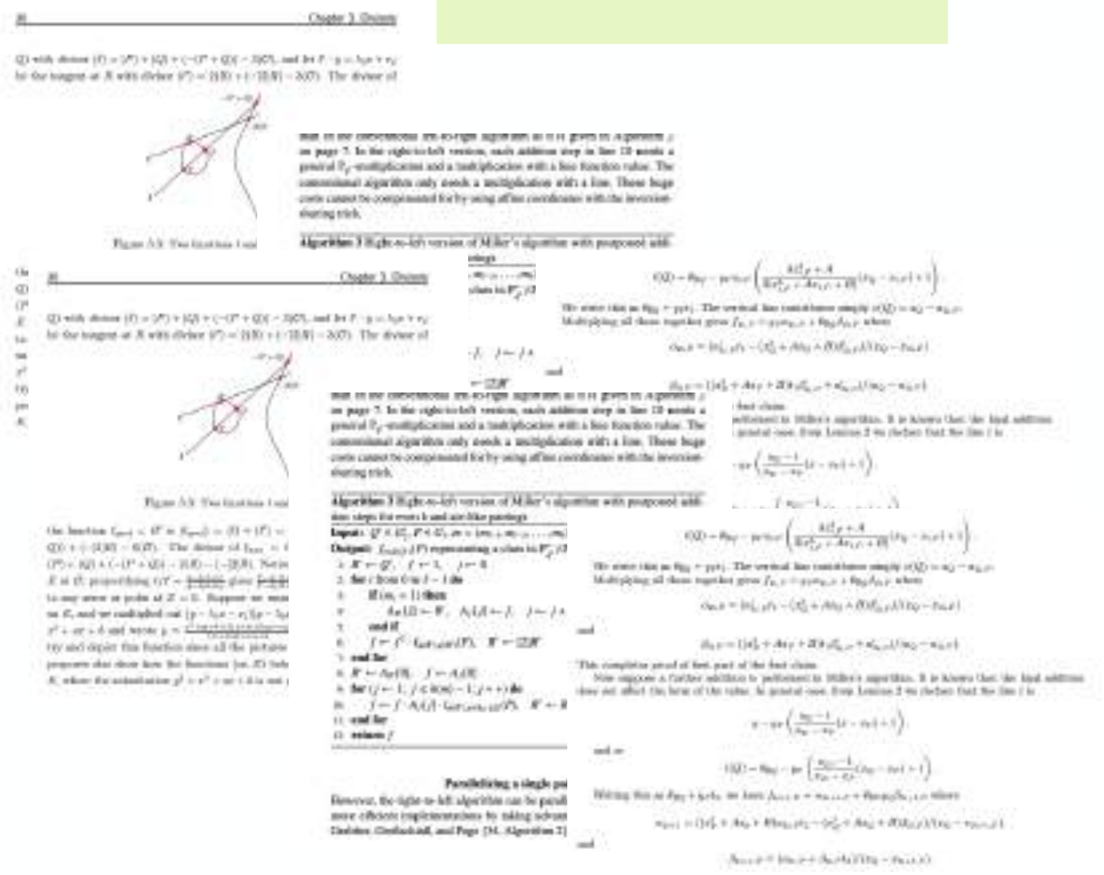

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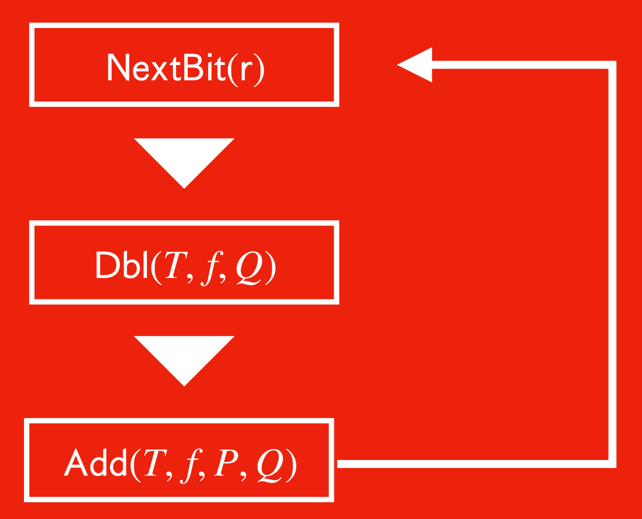

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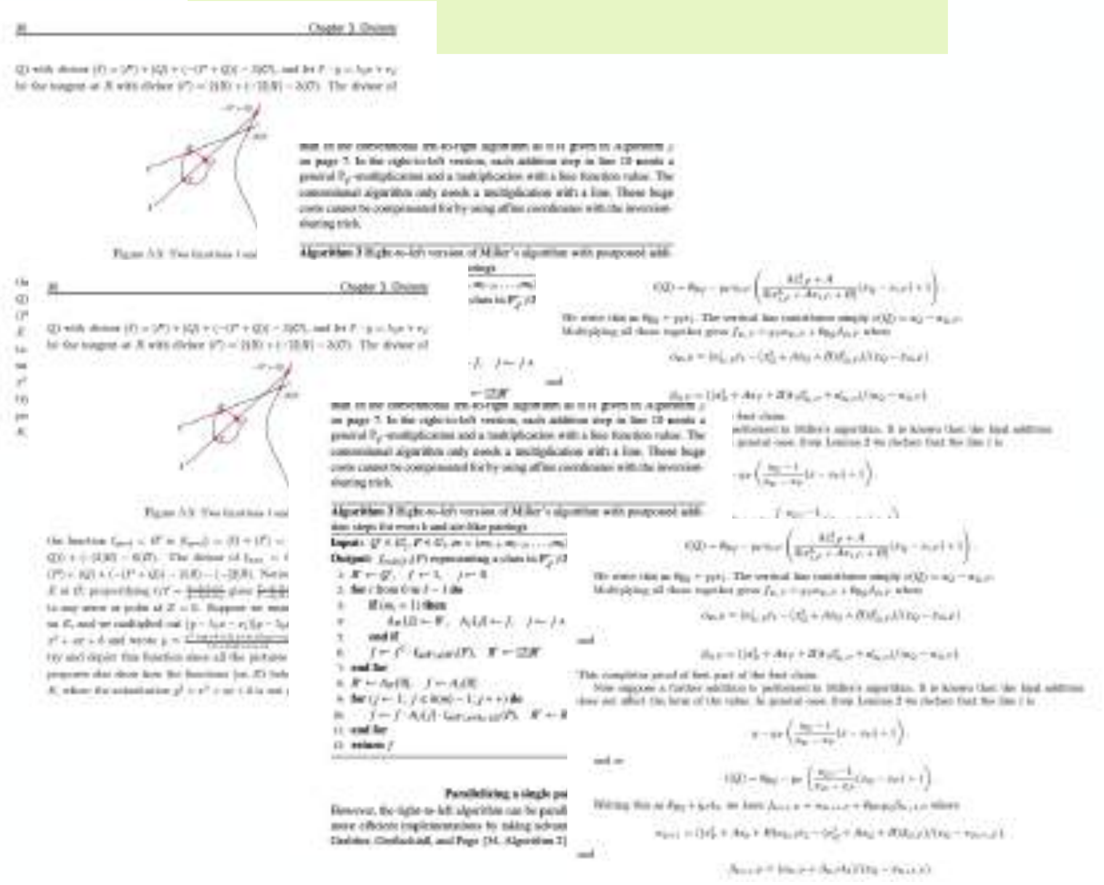

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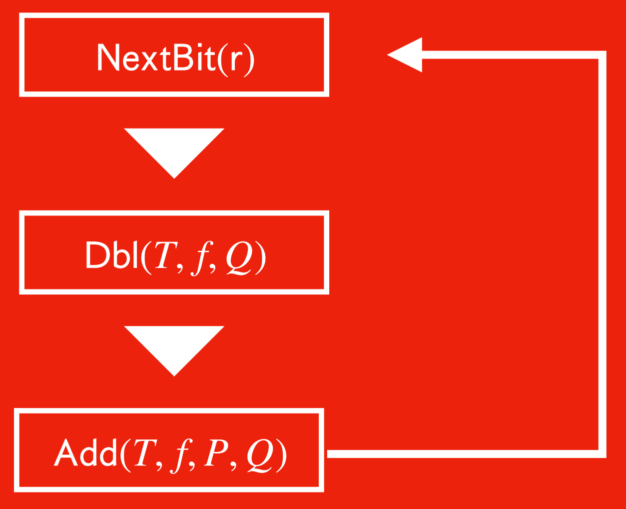

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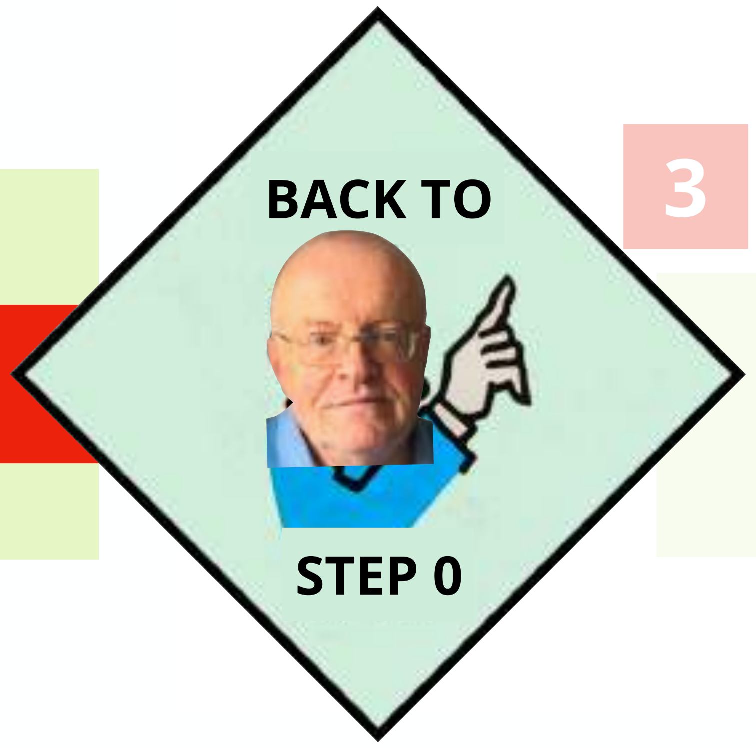

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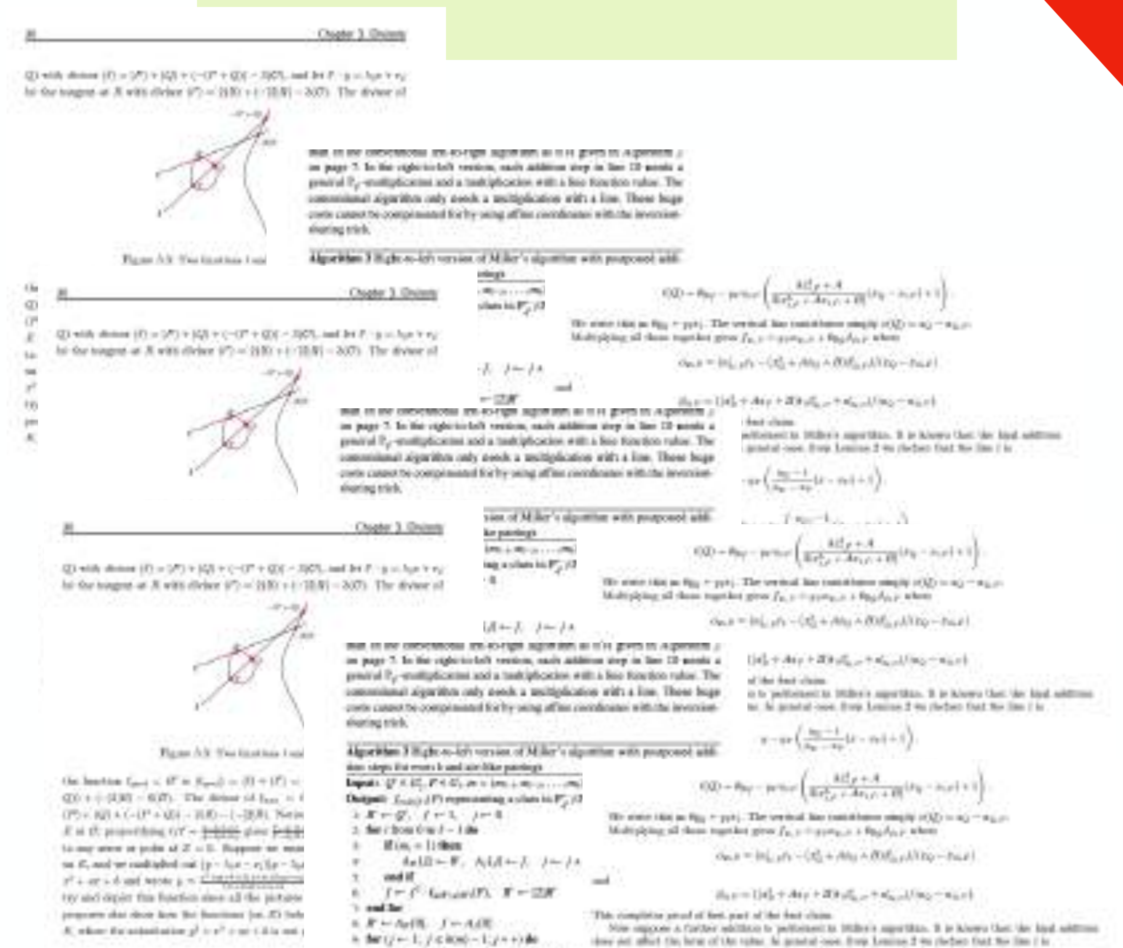

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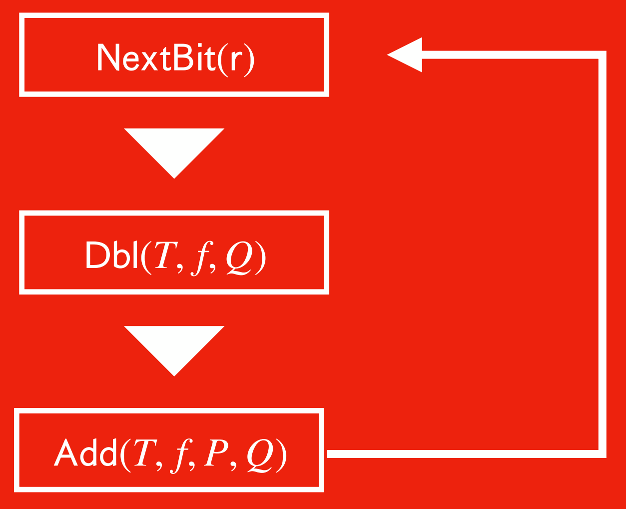

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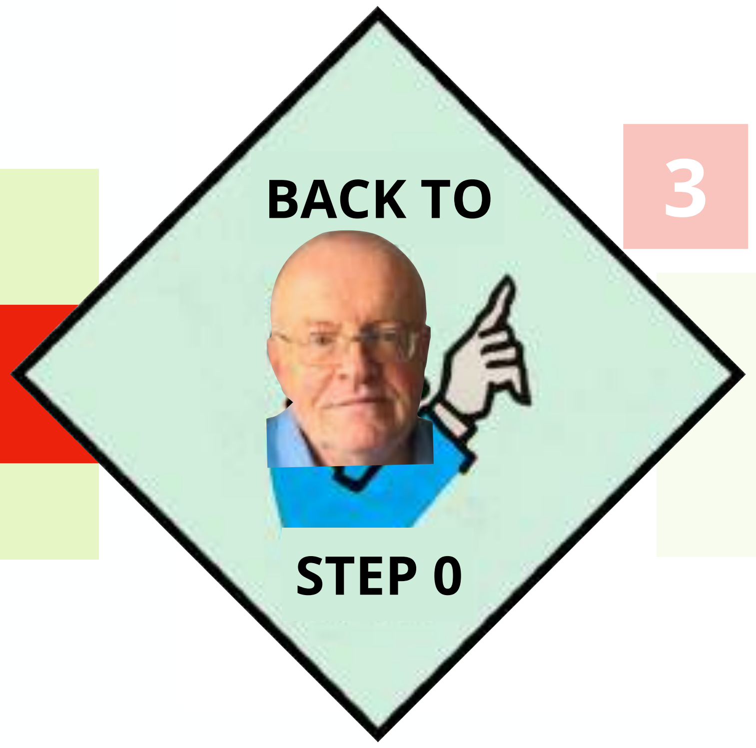

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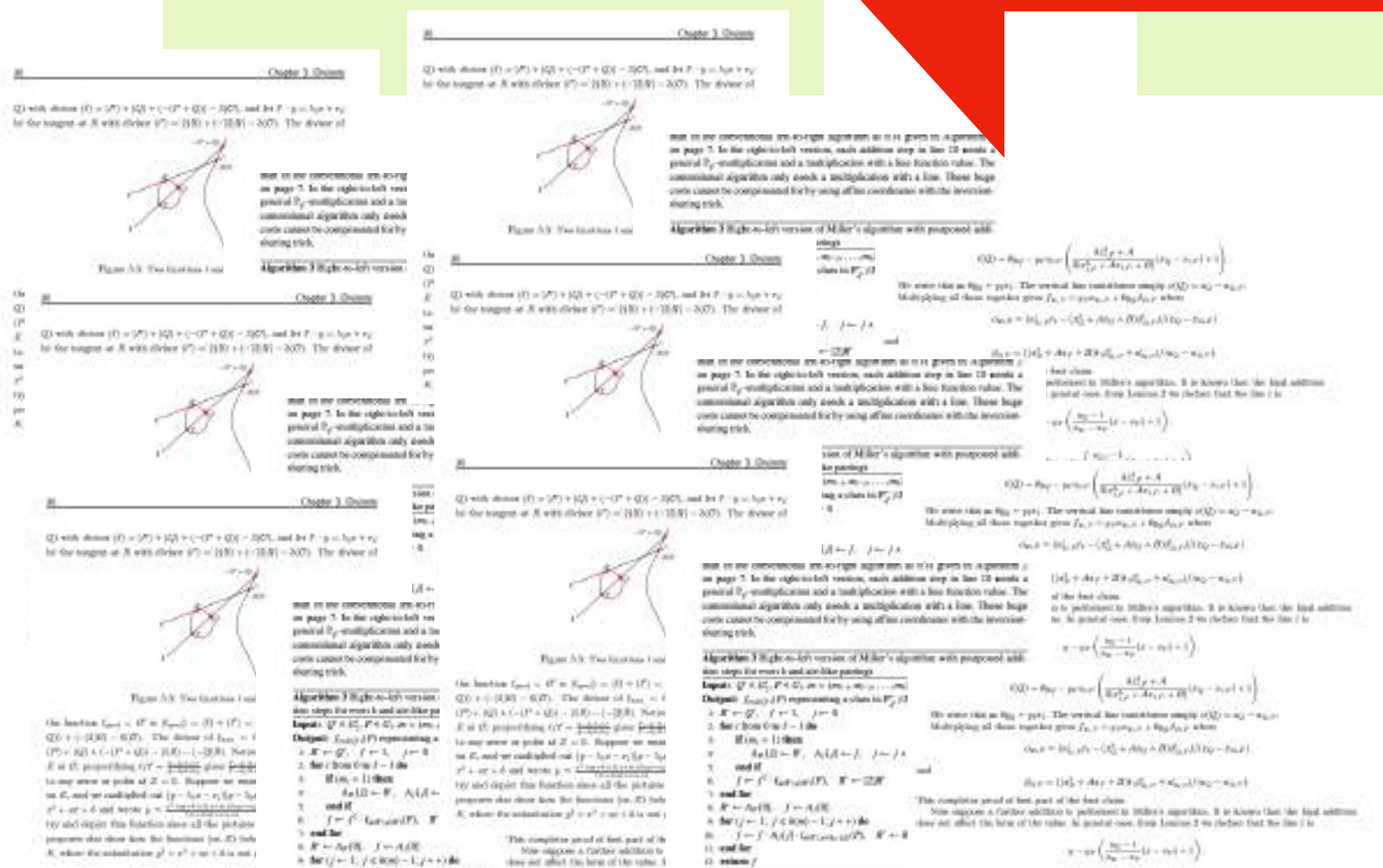

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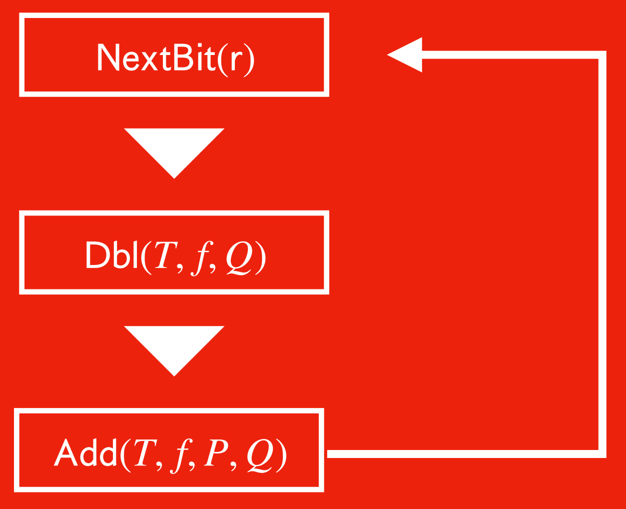

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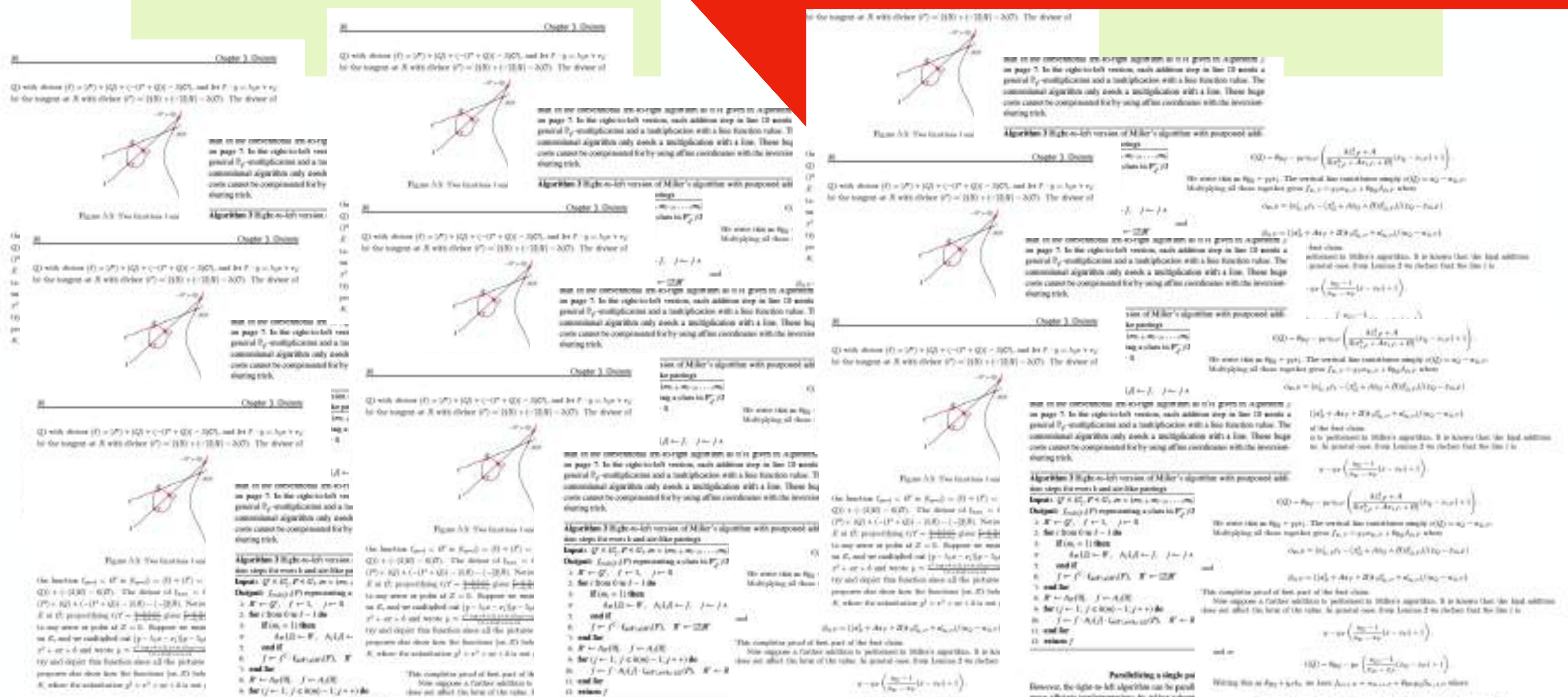

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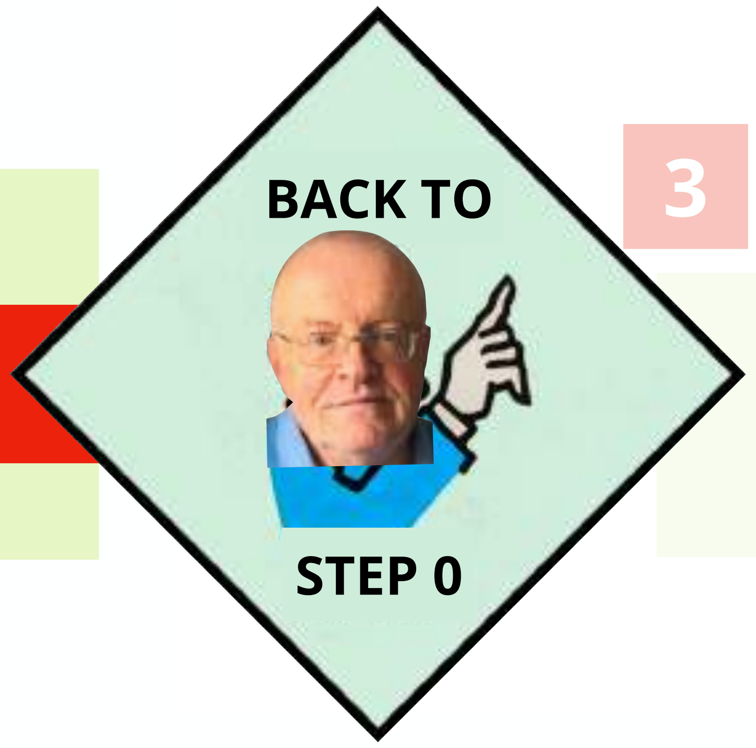

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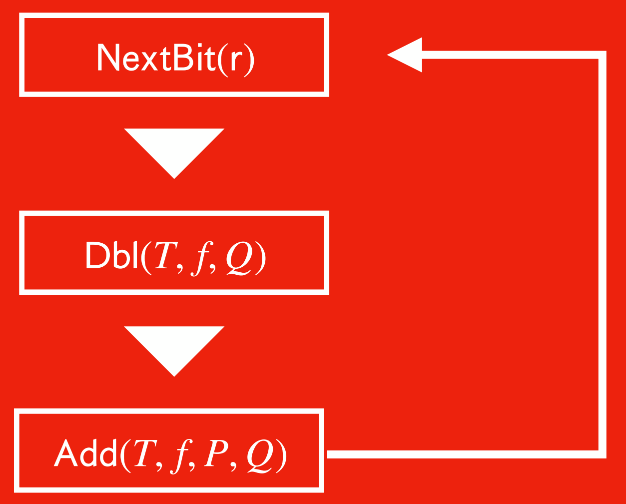

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BACK TO

STEP 0

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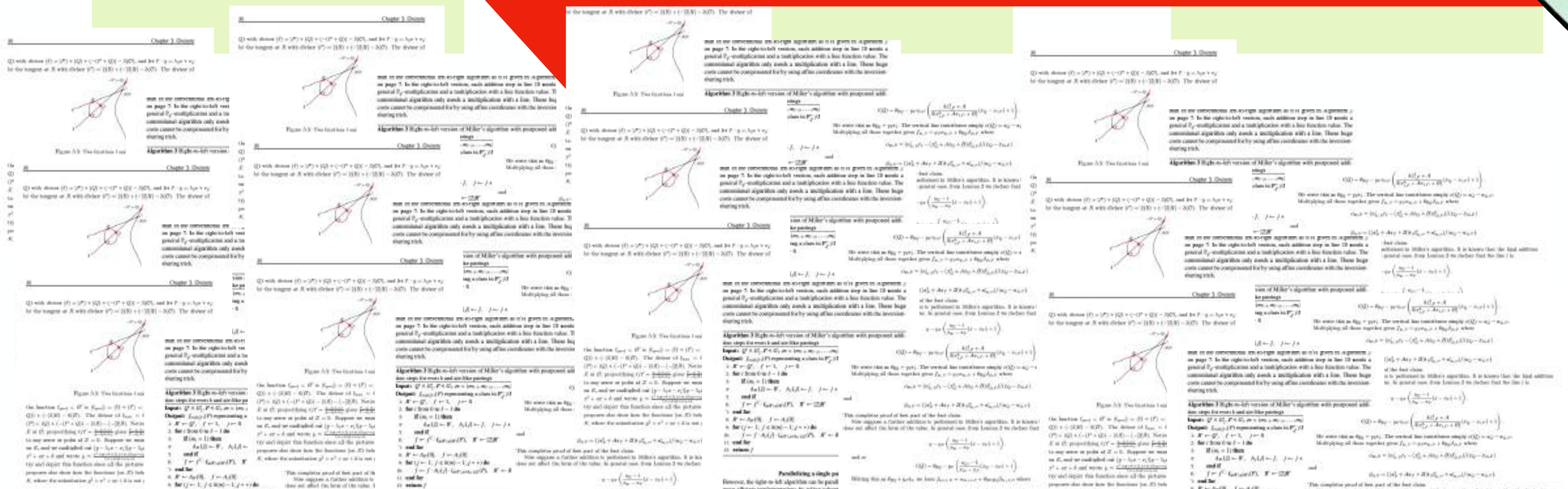

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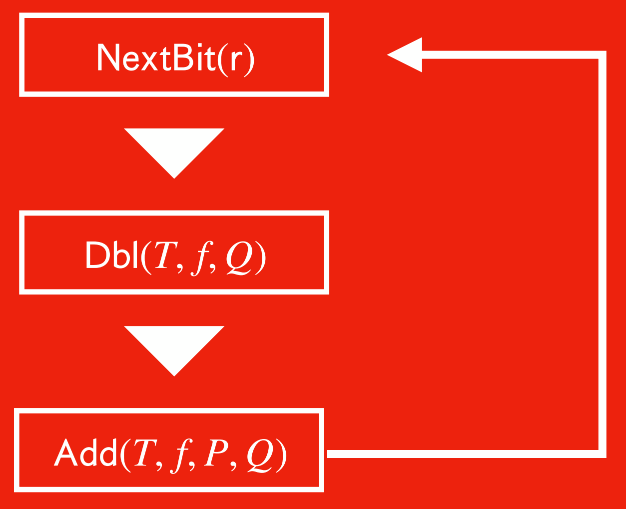

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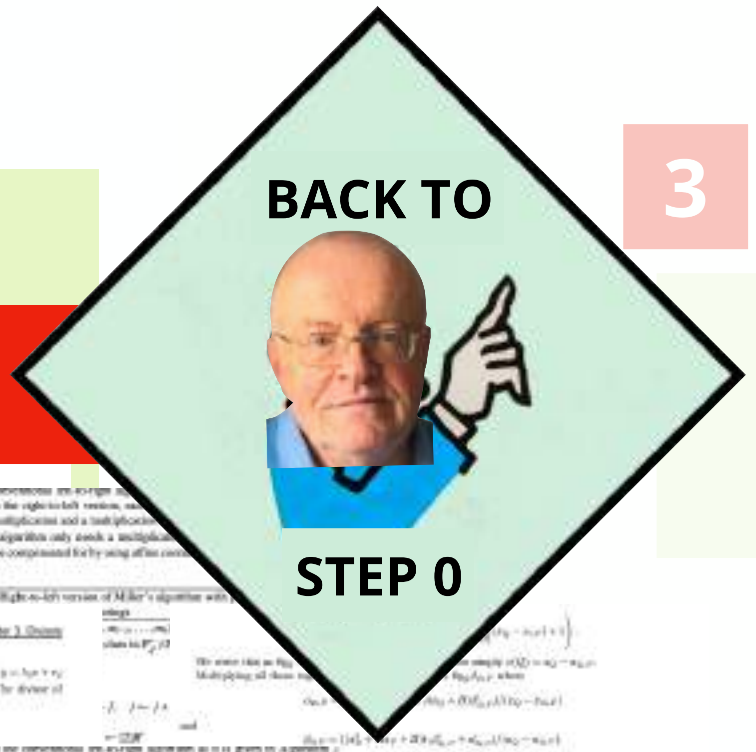

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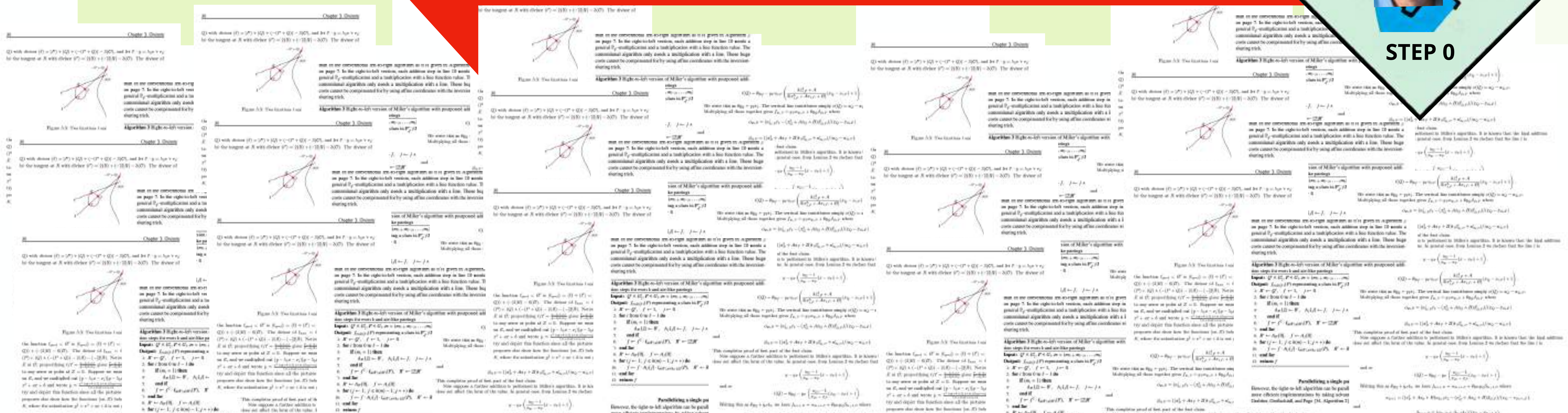

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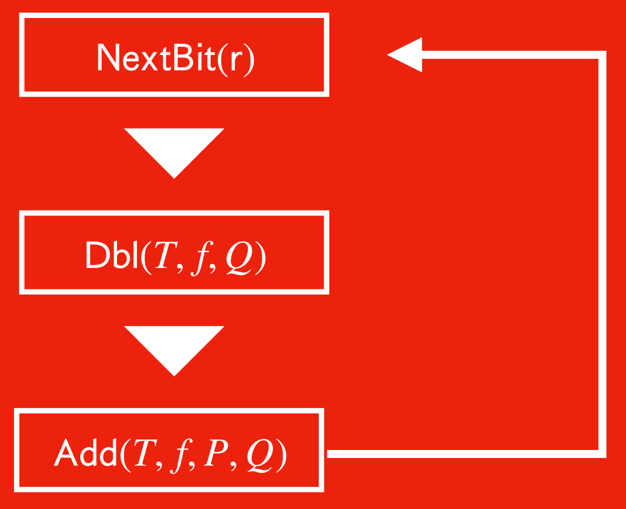

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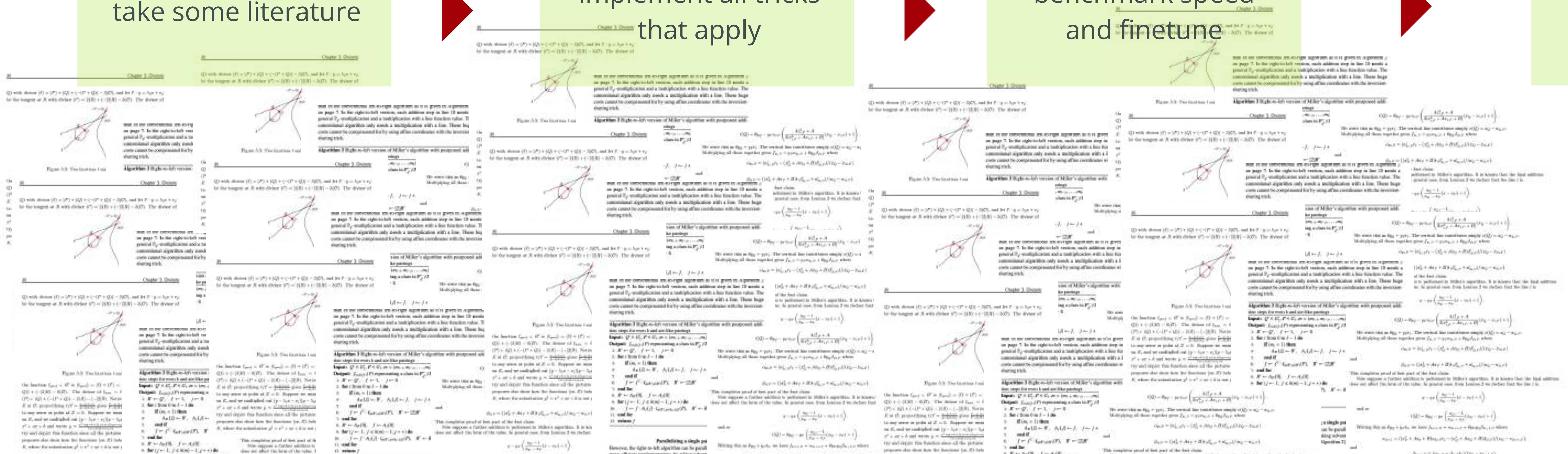

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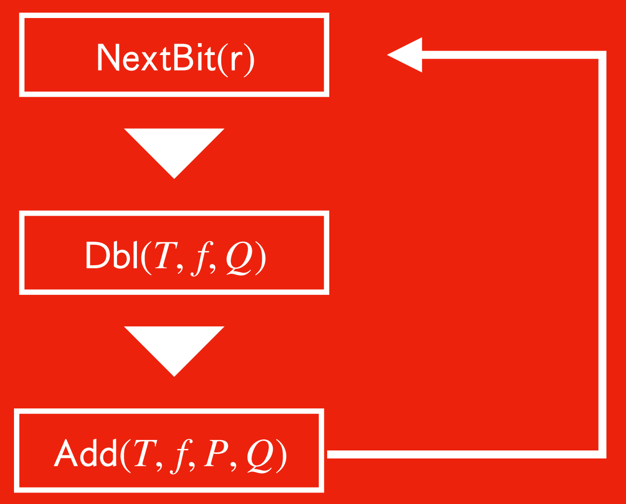

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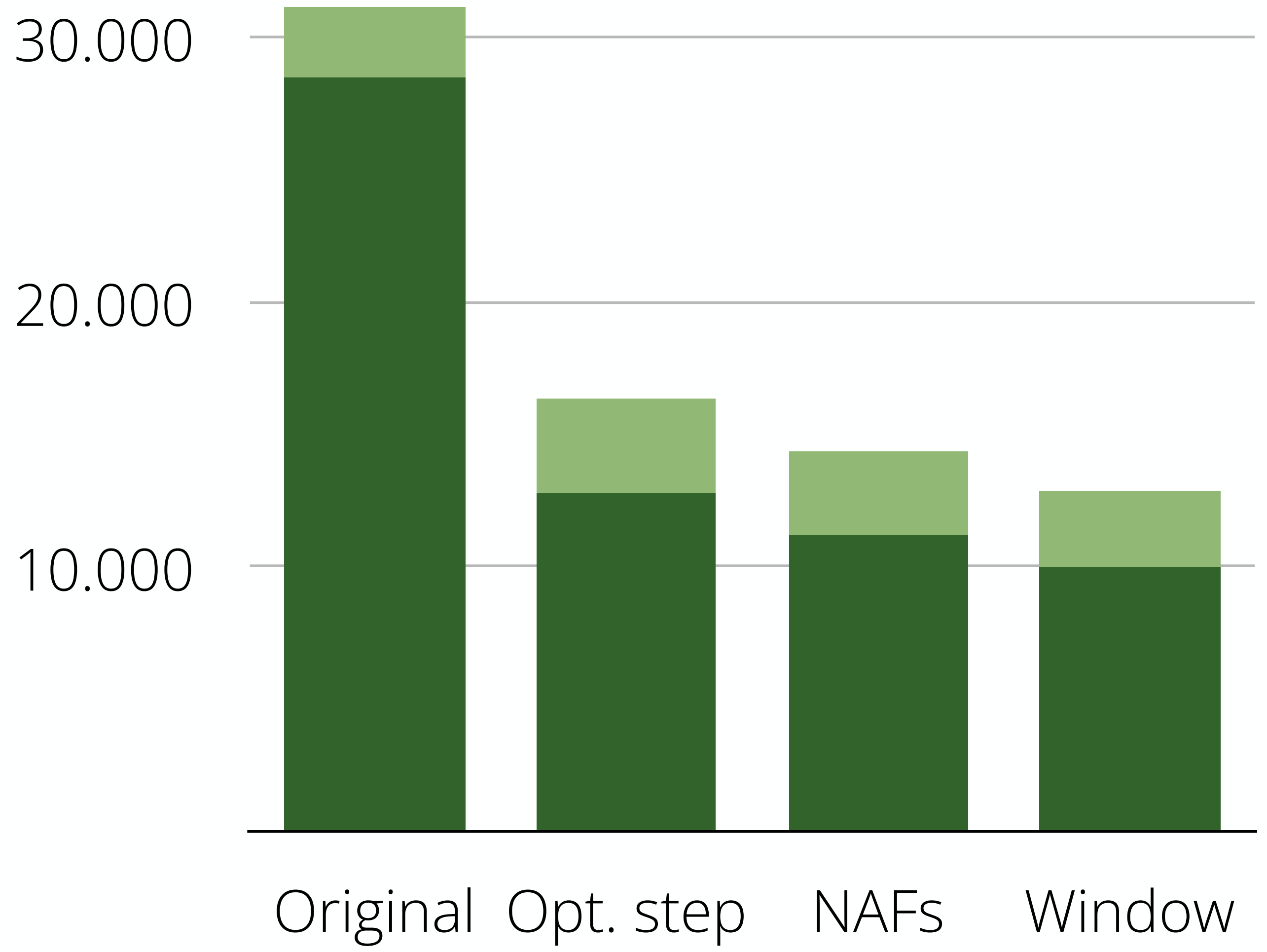

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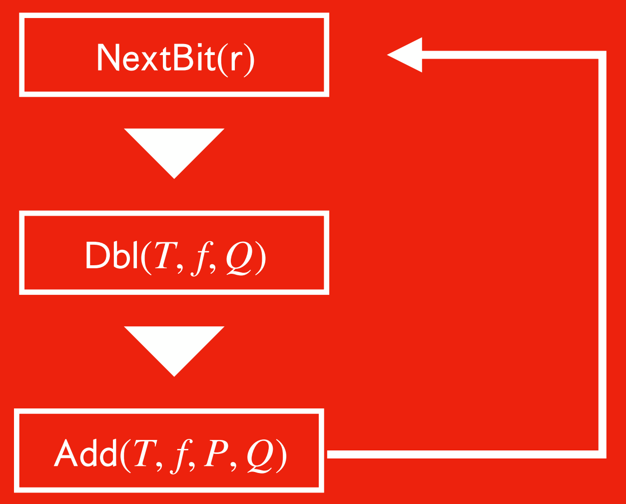

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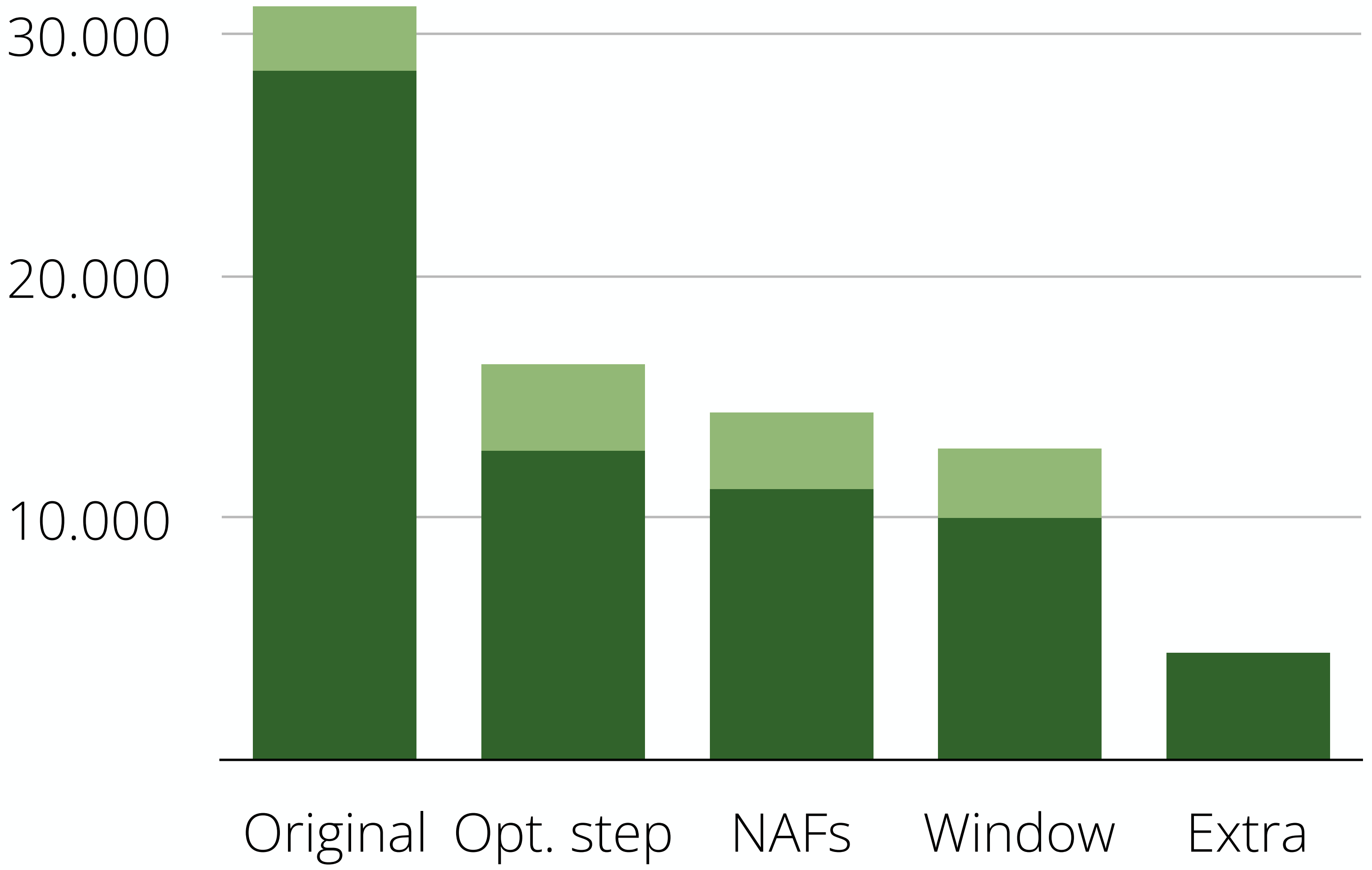


extra pairings

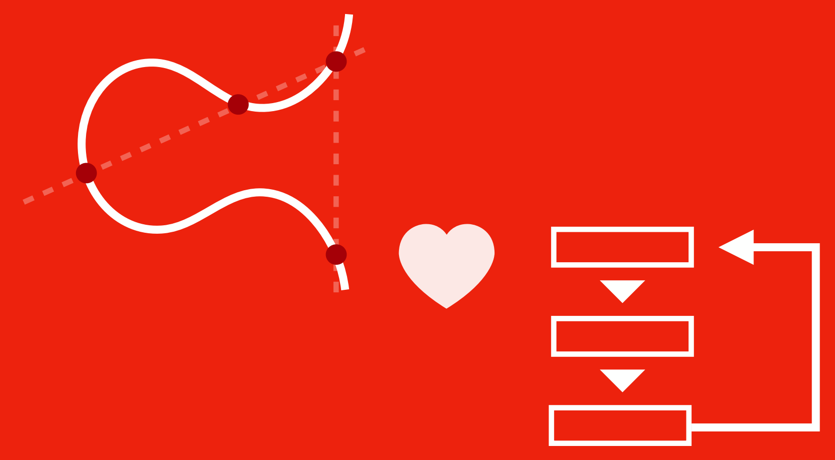
if you have already computed
$e(P, Q_1)$,

it is very efficient to compute
$e(P, Q_2)$

Fast pairings

❤

Isogeny crypto



Applying pairings in isogeny crypto

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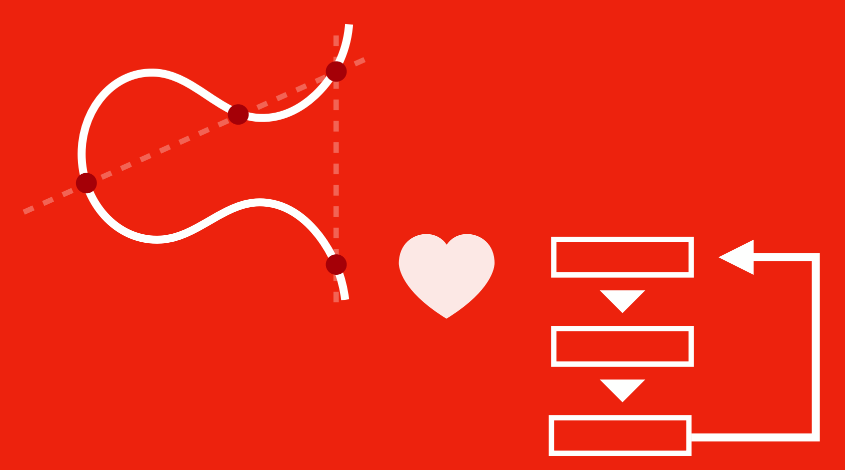

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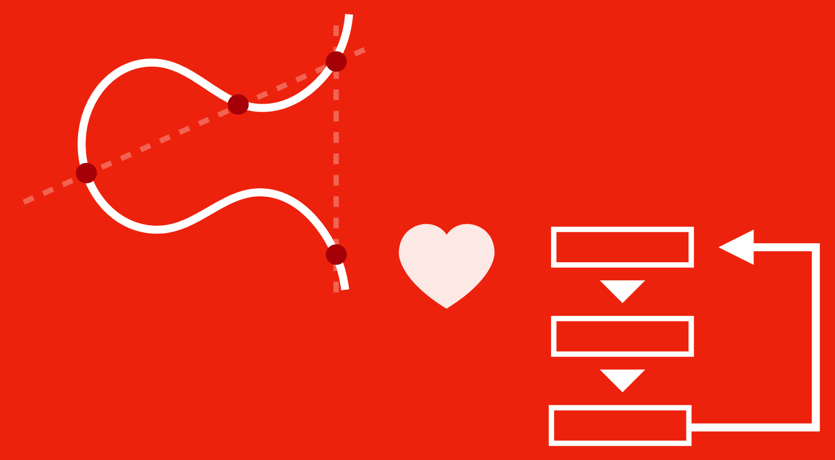

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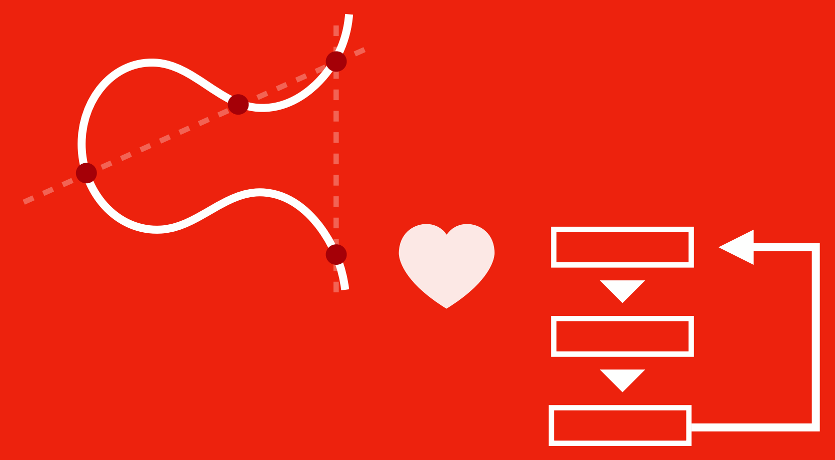

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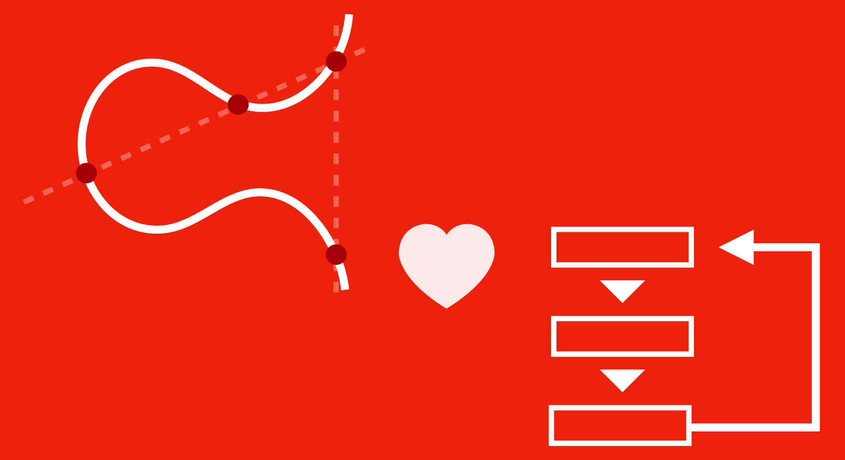

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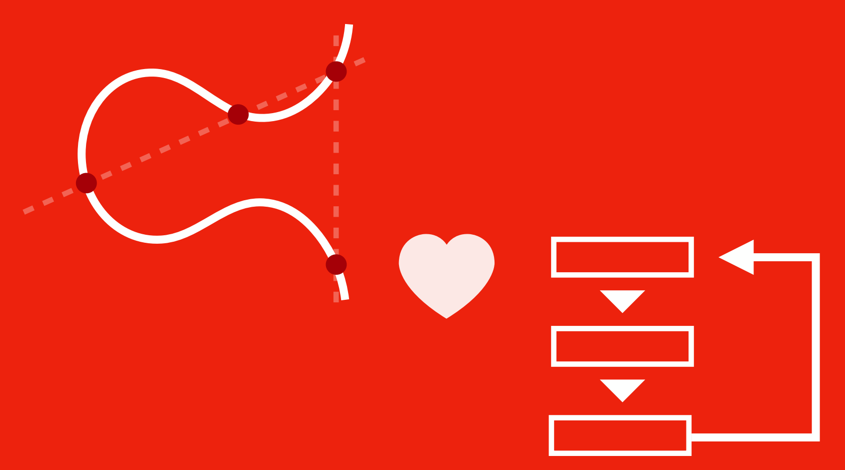

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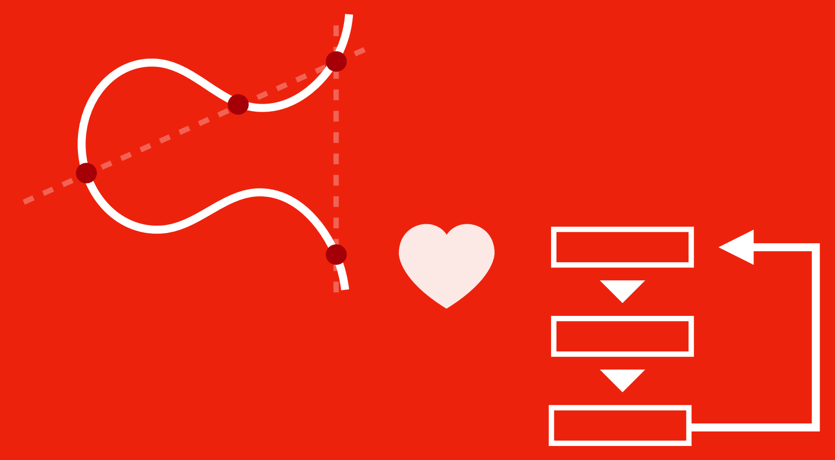

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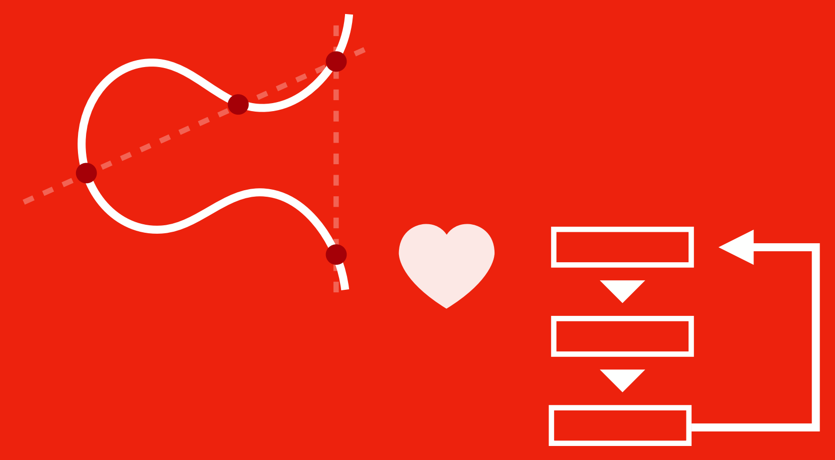

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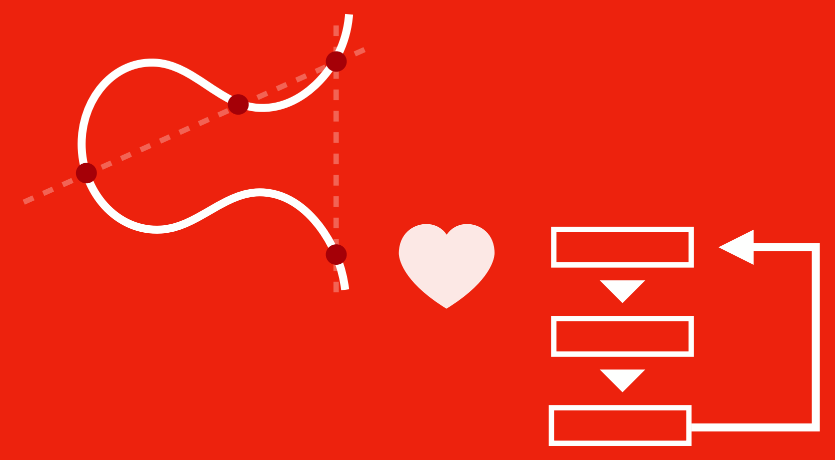

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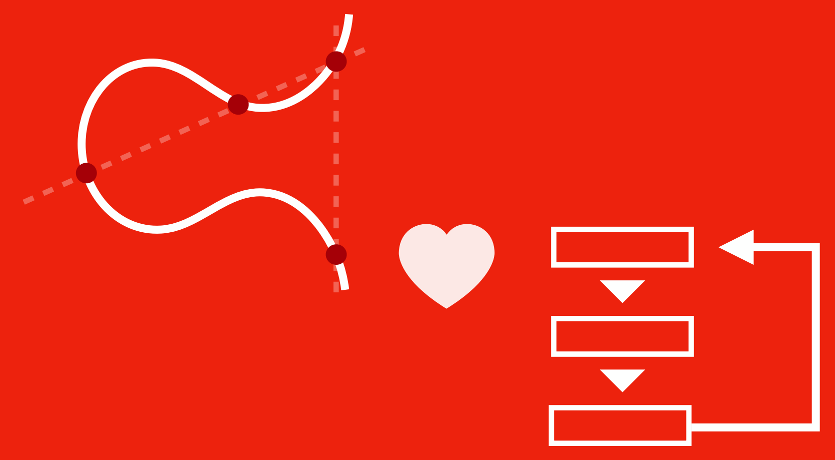

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speedup: -27% compared to CSIDH's

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speedup: +2% compared to Doliskani

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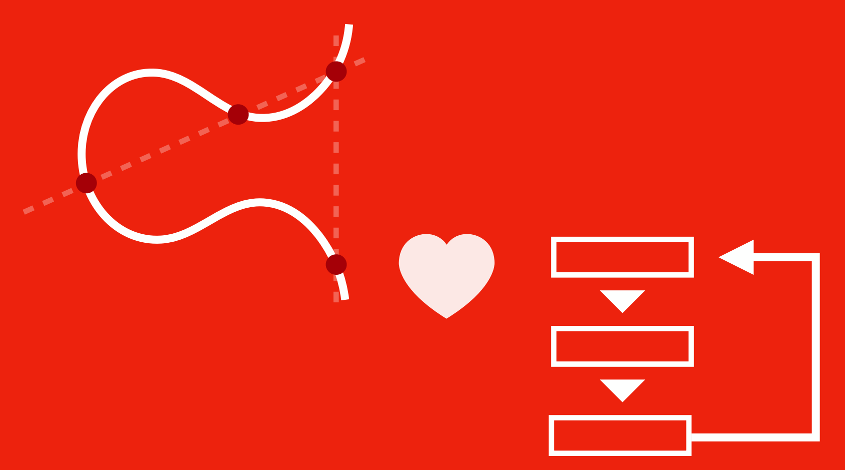
Applying pairings in isogeny crypto

why pairings at all?

scheme maturity

- classical security well understood
- quantum security well understood
- fast, constant-time implementation
- deterministic and dummy-free



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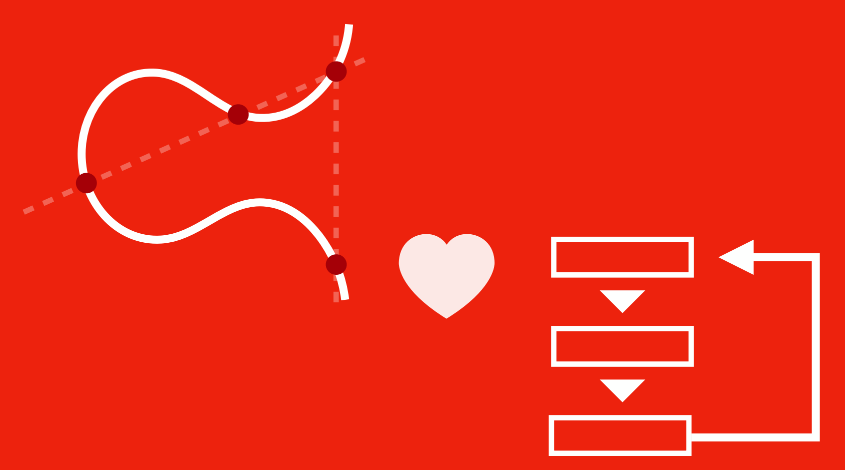

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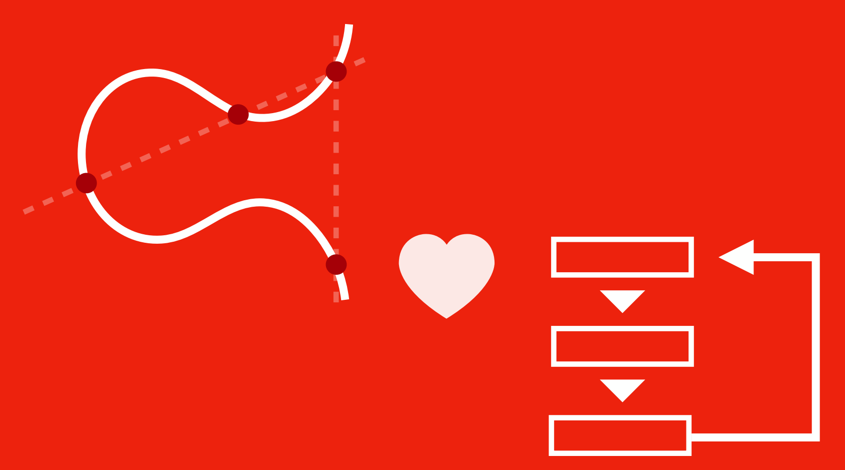

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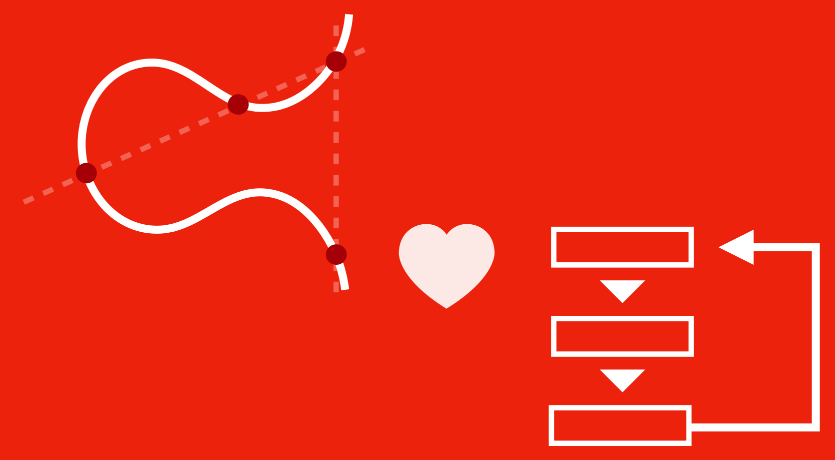

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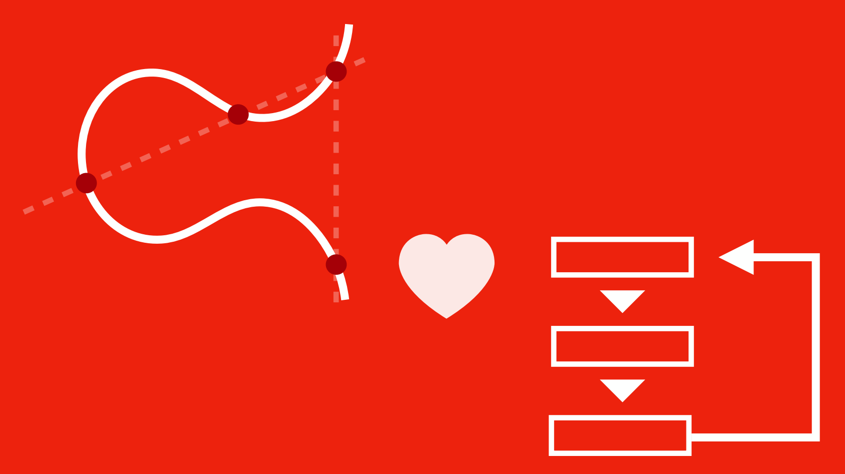
how do we achieve fast high-security CSIDH?

constant-time, deterministic, dummy-free

✗ **previously** → ? **with pairings** → ✓ ***to do***



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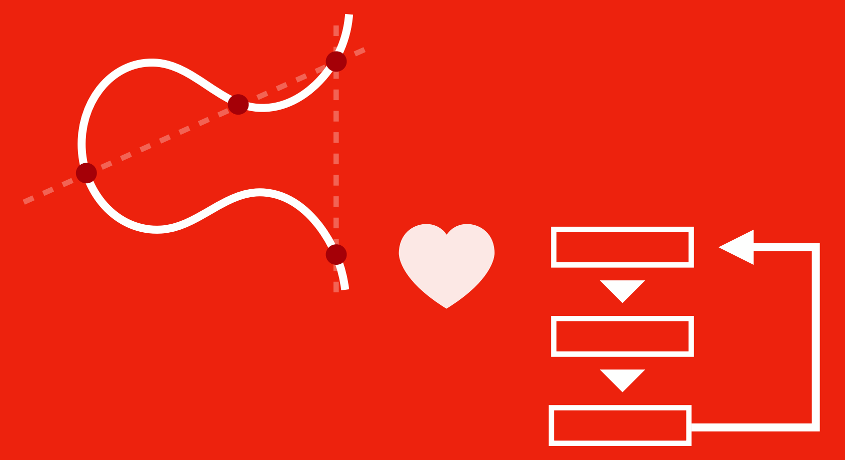

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- removes probability from CTIDH
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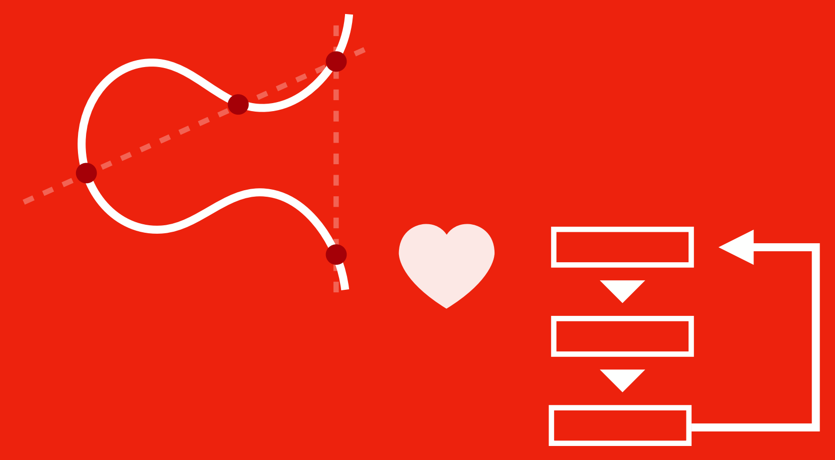

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✓ *to do*

- analyse **optimal** use of torsion
- can we use **faster** torsion finding?
- can improve group action!

Thank you!

Any questions*?

*If not, I have a question for you...

Constant-time Gauss' algorithm?

Finite field world

Q: Given $\mathbb{F}_q$ find generator ζ for $\mathbb{F}_q^*$

Curve world

Given curve E over $\mathbb{F}_p$,
find full torsion point P

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Finite field world

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A:

GAUSS' ALGORITHM

1. Take random $\zeta \in \mathbb{F}_q$, compute $t = \text{Order}(\,\zeta\,)$
2. If $t = q - 1$, **stop**,
3. **else** take random $\beta \in \mathbb{F}_q^*$ and compute $s = \text{Order}(\,\beta\,)$
   a. if $s = q - 1$, **stop**
   b. **else** find coprime $d \mid t$ and $e \mid s$ with $d \cdot e = \text{lcm}(t, s)$
   c. set $\zeta \leftarrow \zeta^{t/d} \cdot \beta^{s/e}$ and $t \leftarrow d \cdot e$ and **repeat** from 2.

Curve world

Given curve E over $\mathbb{F}_p$,
find full torsion point P

Take P and Q,
Compute their torsion.
If P not full torsion,
take right multiple Q
set $P \leftarrow P + Q$ to fill
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